Similarity of Information in Games *

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April 9, 2025

Abstract

What is a reasonable notion for comparing similarity of information across agents in any Bayesian game? At the very least, more similar information across agents should aid coordination. However, it turns out that existing stochastic orders that compare the interdependence of joint distributions do not have this intuitive property. We propose a new class of orders called "Concentration Along Diagonal" (CAD) that compare information similarity in Bayesian games. When information becomes more CAD-similar, each agent believes it is more likely that others have also received the same information. We show that CAD-similarity is *equivalent* to aiding coordination in canonical binary-action coordination games. That is, more CAD-similar information aids coordination in all such games, and if an information change aids coordination in all such games, then it must be more CAD-similar. We apply CAD in other well-known games such as congestion, collective action, or auctions.

Keywords: coordination games, information similarity,

JEL Codes: D82, D83

*This research was supported in part by National Science Foundation award 2417694. We are grateful to Nageeb Ali, Dirk Bergemann, Geoffroy De Clippel, Duarte Goncalves, Nima Haghpanah, Johannes Hörner, Elliot Lipnowski, Michael Mandler, Meg Meyer, Dilip Mookherjee, Stephen Morris, Alessandro Pavan, Ludvig Sinander, Bruno Strulovici, and conference audiences and seminar participants at Bonn University, Brown University, Carnegie Mellon University, Cornell University, Indian Statistical Institute Delhi, National University of Singapore, Ohio State University, Oxford University, Paris School of Economics, Royal Holloway University, Toulouse School of Economics, University of Rochester, University of Surrey, for their helpful comments and suggestions. Basak: Indiana University, Kelley School of Business, email: dbasak@iu.edu; Deb: New York University, email: joyee.deb@nyu.edu, Kuvalekar: University of Essex, email: a.kuvalekar@essex.ac.uk.

1. Introduction

In recent times, we have witnessed fundamental changes in how information is disseminated, driven by innovations like algorithmic targeting of news and personalized content delivery. An important effect of algorithmic targeting is an increased homogenization of information among like-minded individuals: People with similar demographic traits or preferences are exposed to the same content.[1] Especially in strategic interactions, this homogenenization can play a crucial role in fostering coordination. Both public discourse and academic research have highlighted how access to the same information on Twitter enabled coordination precipitating the run on Silicon Valley Bank, or how social media affected participation in mass protests.[2] However, for a systematic inquiry into the effect of changing information similarity on strategic interactions, we first need to answer more foundational questions: How should we even compare the similarity of information structures in such contexts? What kind of information homogenization helps agents coordinate actions when they face strategic uncertainty? This is what we do in our paper.

We study settings with strategic uncertainty, and model agents' private information as a joint distribution over signals. Our main contribution is to propose a novel class of stochastic orders—*Concentration along the Diagonal (CAD) orders*—to compare interdependence of these distributions, and show how they can be used to study the effect of information similarity in strategic settings. CAD orders capture the homogenization narrative: An increase in information similarity in a CAD order simply means that, given their own information, each agent believes that it is now more likely that others have also received information similar to their own.

What is a reasonable way to compare interdependence of joint distributions? Our starting premise is that any economically meaningful order of similarity of information structures in strategic settings should have at least the following feature: If agents have common interests and are engaged in a pure coordination game in the presence of some incomplete information (about the state of the world or others' types), then giving these agents information that is more similar (greater in the chosen order) must surely aid coordination. By aiding coordination, we formally mean that any equilibrium of the game with incomplete information should remain an equilibrium when information becomes more similar. Somewhat surprisingly, existing stochastic orders that could be potentially used to compare similarity—like correlation, supermodular order, positive quadrant dependence etc—do not

[1] A large literature documents the decrease in diversity of content consumption across users as a result of recommendation algorithms. See, for example, Anwar et al. (2024); Aridor et al. (2020); Chaney et al. (2018); Nechushtai et al. (2024); Nguyen et al. (2014) for evidence of homogenization in various contexts from online purchases to news consumption.

[2] See for instance, this article from The Guardian or Cookson et al. (2023); Gam et al. (2023) on bank runs. Cantoni et al. (2019); Enikolopov et al. (2020); Manacorda and Tesei (2020)) study how social media has affected the turnout in mass protests.

satisfy this intuitive normative property.[3] By contrast, the CAD orders we propose not only satisfy this notion of increasing information similarity in strategic settings but, in fact, also *characterize* it: The main results of this paper establish that information similarity increases in a CAD order if and only if the equilibrium set (weakly) expands in a canonical class of binary-action coordination games of incomplete information.

To see why existing orders are not appropriate in strategic settings, we start with a simple example.

Example 1 (Morris and Shin (2003))**:** *Two firms, $i \in \{1, 2\}$ must simultaneously choose whether to make a costly investment in a new technology. If firm i does not invest, it gets 0. If it invests then it gets s_i if the other firm also invests, or $s_i - 1$ if the other firm does not invest. The types s_1, s_2 are $\{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$-valued, and are drawn from a known joint distribution $\mathcal{G}$, and each firm knows its own type.*

	Invest	*Don't*
Invest	s_1, s_2	$s_1 - 1, 0$
Don't	$0, s_2 - 1$	$0, 0$

For a firm i of type $s_i = -\frac{1}{2}$ investing is a dominated strategy, whereas for type $s_i = \frac{3}{2}$ not investing is a dominated strategy. Type $s_i = \frac{1}{2}$ invests if and only if it believes the other firm will invest. Therefore, given joint distribution $\mathcal{G}$, firms with positive type always investing is an equilibrium, if and only if the following condition holds.

$$\frac{1}{2} Prob\left(\mathbf{X}_{-i} \in \left\{\frac{1}{2}, \frac{3}{2}\right\} \middle| \mathbf{X}_i = \frac{1}{2}\right) - \frac{1}{2}\left(1 - Prob\left(\mathbf{X}_{-i} \in \left\{\frac{1}{2}, \frac{3}{2}\right\} \middle| \mathbf{X}_i = \frac{1}{2}\right)\right) \geqslant 0$$

$$\text{or, } Prob\left(\mathbf{X}_{-i} \in \left\{\frac{1}{2}, \frac{3}{2}\right\} \middle| \mathbf{X}_i = \frac{1}{2}\right) \geqslant \frac{1}{2}.$$

What happens with increased homogenization of information? Since this is a coordination game of incomplete information, we should expect reasonably that if the firms' signals are more similar, coordination should be easier i.e., if it is an equilibrium for a firm with a positive type to invest under $\mathcal{G}$, then it should still be true if information became "more similar." But, for the above condition to continue to hold, more similar information must mean specifically that $\mathbf{X}_i = \frac{1}{2}$ assigns a (weakly) higher probability that $\mathbf{X}_{-i} \in \{\frac{1}{2}, \frac{3}{2}\}$. Interestingly, increasing correlation between signals or increases in the supermodular order, positive quadrant dependence order or concordance order do not guarantee this. In fact increasing interdependence in these standard orders can result in an agent with signal $\frac{1}{2}$ assigning a strictly *lower* probability to the other agent receiving a positive signal.

To see this, consider two joint distributions in Figure 1. $\mathcal{F}$ and $\mathcal{G}$ represent

[3] There is a large literature in statistics and economics on stochastic orders. See Meyer and Strulovici (2012, 2015); Meyer (1990); Müller and Scarsini (2000); Müller and Stoyan (2002) for instance.

two joint distributions over $\{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\} \times \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$. For example, the -2α means that the probability mass on $(\frac{1}{2}, \frac{1}{2})$ under $\mathcal{F}$ is 2α less than under $\mathcal{G}$. Note that $\mathcal{F}$ and $\mathcal{G}$ have the same marginal distributions, and it is easy to verify that $\mathcal{F}$ has a higher correlation than $\mathcal{G}$, and is also greater than $\mathcal{G}$ in the supermodular order, positive quadrant dependence order or concordance orders, for any positive and feasible α.

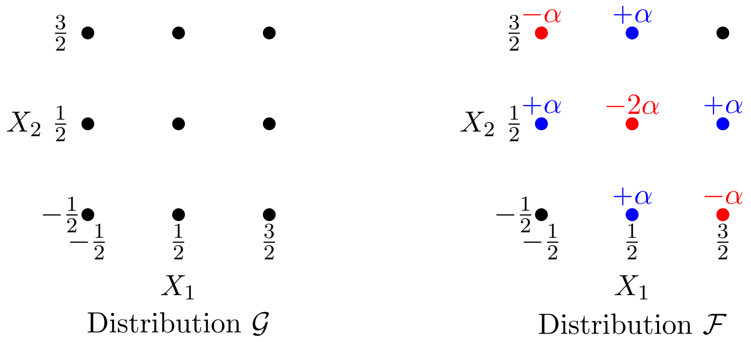


Figure 1: Increasing interdependence according to existing orders ($\alpha > 0$).

Start with a joint distribution $\mathcal{G}$ under which the equilibrium condition holds, and consider α such that for all i

$$\alpha \geqslant Prob\left(\mathbf{X}_{-i} \in \left\{\frac{1}{2}, \frac{3}{2}\right\} \middle| \mathbf{X}_i = \frac{1}{2}\right) - \frac{1}{2}.$$

The strategy profile in which each firm with a positive type invests is an equilibrium under $\mathcal{G}$, but not under $\mathcal{F}$, the more "similar" distribution! An increase in similarity, as measured by these orders seems to hinder coordination even in this simple coordination game. Why might this be? Intuitively, we think more similar information should aid coordination in this example, because conditional on their own private signal being positive, a firm should now think that it more likely that the other firm has also seen a positive signal, and this in turn, should make it easier for them to behave similarly (invest). It turns out that an increase in correlation or the stochastic orders discussed above increase the *joint probability* of agents receiving similar information, but do not necessarily affect agents' *conditional beliefs* in the natural way that drives them to coordinate.

This example highlights that an order of information similarity in strategic settings should compare conditional beliefs. Our new class of Concentration along a Diagonal (CAD) orders does exactly this. An increase in a CAD order means that *conditional on receiving information*, each agent believes that it is now more likely that others have also received the same information.

Our main results establish that increasing similarity in the CAD order is equivalent to aiding coordination. Informally stated, our results are as follows:

Theorems: $\mathcal{F}$ is more similar than $\mathcal{G}$ in the CAD order if and only if the equilibrium set under $\mathcal{F}$ is larger than the equilibrium set under $\mathcal{G}$

for a canonical class of binary action coordination games.[4]

Importantly, this is a characterization. Increasing information similarity in the CAD order expands the equilibrium set in coordination games. And conversely, if $\mathcal{F}$ is not more similar than $\mathcal{G}$ in the CAD order then we can find a canonical coordination game and a strategy profile that constitutes an equilibrium of that game under $\mathcal{G}$ but not under $\mathcal{F}$.

Formally, we consider binary-action private-value coordination games with multiple players, in which a player's payoff depends on the aggregate actions of others, and on her own private type. The payoff difference between taking the two actions is of the form

$$d(\mathbf{A_{-i}}, s) = \alpha(s) + \beta(s)h(\mathbf{A_{-i}}),$$

where $\mathbf{A_{-i}}$ is the aggregate action of other players, s is the player's private signal and $h(\cdot)$ is increasing, and $\beta(\cdot) \geqslant 0$. So, we have a setting of strategic complementarity. Several canonical examples from the literature share this payoff structure (see Morris and Shin, 2002, 2004, for example). Each player receives a private signal (referred to as her type) before she takes action. To fix ideas, one can think about N firms choosing whether or not to invest in a new technology. Each firm's payoff from investing increases with the aggregate investment and her type that captures the value of the technology to her.

We define an information structure to be a joint distribution of the players' private signals, and introduce three different CAD orders to compare these joint distributions. Our first notion is called *Weak concentration along the diagonal (wCAD)*. We say an information structure $\mathcal{F}$ is more similar than $\mathcal{G}$, or greater in the weak CAD order, if any agent i believes, conditional on her realized signal s, that it is more likely under $\mathcal{F}$ than $\mathcal{G}$, that any other agent j also has exactly the same signal s. There is an equivalent formulation in terms of the number of other agents with the same signal: In particular, $\mathcal{F}$ is greater than $\mathcal{G}$ in the wCAD order means that, conditional on an agent i getting signal s, the *expected number of other agents* with the same signal s is higher under $\mathcal{F}$ than $\mathcal{G}$. It turns out that wCAD is equivalent to an order proposed by Meyer (1990).

Our first main result, Theorem 1, shows that increasing similarity of information in the wCAD order is in fact equivalent to expanding the set of equilibria in this canonical class of private-value coordination games when $h(\cdot)$ is affine.

Our second result goes from affine and increasing $h(\cdot)$ to simply increasing $h(\cdot)$. We define a stronger notion of similarity that we call *Strong concentration along the diagonal (sCAD)*. We say information structure $\mathcal{F}$ is more similar than $\mathcal{G}$, or greater in the strong CAD order, if any agent i believes, conditional on her realized signal s, that the number of other agents with the a similar signal (signal in set K

[4] Arguably there are many ways of formalizing the notion of aiding coordination. In Section 3 we discuss why the equilibrium set inclusion criterion is a natural one in our setting.

that contains s) is higher in the sense of *first-order stochastic dominance* under $\mathcal{F}$ than $\mathcal{G}$. Under wCAD, this ranking was only in terms of expectation. We show, analogous to our first result that increasing similarity of information in the sCAD order is equivalent to expanding the set of equilibria in this class of private-value coordination games for any increasing $h(\cdot)$.

Notice that an increase in similarity, as measured by the weak (or strong) CAD, requires that any two agents see *exactly the same signal* with a higher probability. This may be too strong in some contexts, making the orders quite incomplete. Suppose that the signals are ordered. Indeed, two information structures $\mathcal{F}$ and $\mathcal{G}$ are not comparable even if the probability of the signals being very close in value was higher under $\mathcal{F}$, and the probability of signals being very different in value was lower under $\mathcal{F}$. This motivates our definition of another order, *contour-set CAD*, which uses the order structure of the signal set. Roughly speaking, increased similarity in the countour-set CAD order means that the signals are close to each other in value with a higher probability.

We then study a class of common-value affine coordination games that are widely studied in the global games literature. In these settings, there is an unknown but common state that determines players' payoffs, and players get private signals about the state before they choose whether to take an action or not. In these applications, it is common to restrict attention to cutoff equilibria, where a player acts if and only if her type is high enough. We show that, increasing similarity in the contour-CAD order expands the set of cutoff equilibria. The converse is also true under an additional regularity condition.

Finally, we discuss how the CAD orders can be used to derive economically meaningful predictions in settings much beyond the canonical binary-action coordination games with finite signals. We consider settings with strategic substitutability like congestion games, and show that, analogous to our main results, more CAD-similar information shrinks the set of symmetric pure strategy Bayesian Nash equilibria in congestion games. We apply CAD in an auction setting to show that the revenue from a second-price auction increases when players' valuations become more similar in the weak CAD order. We also present extensions to collective action games and settings with non-exchangeable signal distributions. Finally, we use CAD to study the effect of increased information similarity in a standard global game with a continuum of states, an improper prior over states and private Gaussian signals, thus showing the applicability of CAD beyond finite environments.

1.1. Related Literature

Our paper relates to a literature in economics and statistics that proposes various orders to compare interdependence of random variables, (see Meyer and Strulovici, 2012, 2015; Meyer, 1990; Müller and Stoyan, 2002, for instance). Meyer and Strulovici (2012) and Meyer and Strulovici (2015) illustrate the usefulness of various existing orders, and particularly the supermodular order, for economic applications. Epstein and Tanny (1980) provide a behavioral foundation based on preferences for correlation for the supermodular order in the case of bivariate random variables.[5] However, most of the existing dependence orders do not compare the *conditional* belief distributions that arise in strategic settings with incomplete information.[6]

We contribute more broadly to research on games of incomplete information. A large literature that dates back at least to Hirshleifer (1971) studies how exogenous changes in the information environment impacts behavior in games. More recently, Morris and Shin (2002), Angeletos and Pavan (2007), Bergemann and Morris (2013), Jensen (2018), and Mekonnen and Vizcaíno (2022) have studied this in a class of games similar to ours where information is dispersed among agents. These studies focus on the value of new information, whereas our setup has no new information. Instead, the information becomes more or less similar. When there is a common state, as is usually the case in the above mentioned papers, similarity of information affects the agents' strategic uncertainty but not their fundamental uncertainty. Unlike in the global games literature, our focus is not on equilibrium uniqueness, but rather we show how the set of equilibrium changes with similarity of information. Gossner (2000), Cherry and Smith (2012), and Bergemann and Morris (2016) have also proposed raking of information structures based on equilibrium set inclusion.

We consider the effect of changing interdependence of multivariate random variables while keeping fixed the marginal distributions. Clemen and Winkler (1985) and Cheng and Borgers (2024) study how such changes impact the value of information and show that informational diversity may be valuable. de Oliveira et al. (2023) consider an environment with known marginal distributions but unknown joint distribution to obtain the robustly optimal policy in a class of decision problems. Cripps et al. (2008) and Awaya and Krishna (2025) study the effect of the interdependence of signals on common learning. Awaya and Krishna (2025) show that when agents receive information over multiple periods, then more interdependence of information within period obstructs common learning. This happens because increasing interdependence within a period does not imply that the signals across multiple periods become more similar.[7]

2. Notions of Information Similarity

One of our main contributions is to propose orders to compare the similarity of multivariate distributions. Since we are interested in studying similarity in

[5] Strictly speaking, they focus on the lower orthant order. But, it is equivalent to the supermodular order in two dimensions.

[6] It is worth mentioning that the CAD orders satisfy all the desirable properties of stochastic orders of interdependence proposed by Joe (1997).

[7] See Online Appendix Section B.1 for a more detailed discussion.

strategic settings with incomplete information and multiple agents we define an information structure to be a joint distribution of agents' private signals or types.[8] We define three orders of similarity of information structures. The notions are motivated fundamentally by the idea that more similar information means that conditional on observing some private signal s, a player believes any other player is more likely to have observed the exactly the same signal s. This motivates the name for our class of orders "Concentration along a Diagonal."

Let $\mathcal{S}$ be a fixed, finite set of types. Let $\vec{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N)$ denote the $\mathcal{S}^N-$valued random variable that represents the types of N agents. We assume that $\vec{\mathbf{X}}$ is distributed according to some exchangeable distribution $\mathcal{F}$. For any i and $j \neq i$, and any $K \subset \mathcal{S}$, we define $\mathcal{F}_s(\cdot) \in \Delta(\mathcal{S})$ by

$$\mathcal{F}_s(K) := Prob(\mathbf{X}_j \in K | \mathbf{X}_i = s).$$

In particular, $\mathcal{F}_s(s')$ means $\mathcal{F}_s(\{\mathbf{X}_j = s'\})$. Exchangeability of $\mathcal{F}$ implies that we need not index these conditional distributions with player identities.

2.1. Weak Concentration along a Diagonal

Definition 1 (Weak Concentration along a Diagonal): *We say that $\mathcal{F}$ has a "weakly higher concentration along a diagonal" than $\mathcal{G}$, or $\mathcal{F}$* ***is wCAD higher than*** *$\mathcal{G}$, denoted by $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$, if,*

1. *$\mathcal{F}$ and $\mathcal{G}$ have the same marginal distributions. And,*
2. *For all $s \in \mathcal{S}$, and any i, j*
   (a) *$\mathcal{F}_s(s) \geqslant \mathcal{G}_s(s)$, and*
   (b) *$\mathcal{F}_s(s') \leqslant \mathcal{G}_s(s')$ whenever $s' \neq s$.*

If $\mathbf{X} \sim \mathcal{F}$ and $\mathbf{Y} \sim \mathcal{G}$ with $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$, we say $\mathbf{X} \geqslant_{wCAD} \mathbf{Y}$, i.e., we use $\mathbf{X} \geqslant_{wCAD} \mathbf{Y}$ and $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$ interchangeably.

In words, when information becomes more similar in the weak CAD order, any agent i believes, conditional on being realized type s, that it is more likely that any other agent j is also of exactly the same type s. It is worth noting that verifying whether two distributions are ranked in this order is not computationally hard. For exchangeable distributions, the complexity is $\mathcal{O}(|\mathcal{S}|^2)$. The weak CAD order is equivalent to the orders in Meyer (1990) (see Proposition 1 and 5 from Meyer, 1990).

To gain some more intuition about the wCAD order, we provide some equivalent formulations of the wCAD order in the lemma below. It is almost immediate that an increase in the wCAD order is equivalent to requiring that, conditional on being of type s, any agent i assigns higher probability to another agent j having type in *any subset* of types that includes type s. Another way to formulate the order is in

[8] Henceforth, we will use the terms "types" and "signals" interchangeably.

terms of the number of other agents who have the same type as a given agent i. An increase in similarity in the weak CAD order means that, conditional on any agent i being of type s, the expected number of other agents with type in any subset of types that includes s is higher. We state this formally below. The proof is in the appendix. Define, for any agent i, for any $K \subseteq \mathcal{S}$,

$$\mathbf{I}(K) := \sum_{j \neq i} \mathbb{1}_{\mathbf{X}_j \in K}$$

$\mathbf{I}(K)$ counts the number of players other than player i with realized type in set K. $\mathbf{I}(K)$ is a $\{0, 1, \ldots, N-1\}$-valued random variable. Let $\mathbf{H}_{\mathcal{F}}^{s,K}$ denote the CDF of $\mathbf{I}(K)$ conditional on $\mathbf{X}_i = s$, and $\mathbf{X}$ is distributed according to $\mathcal{F}$.

Lemma 1: *Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{S}^N$-valued, exchangeable random variables with distributions $\mathcal{F}$ and $\mathcal{G}$ respectively. Moreover, $\mathcal{F}$ and $\mathcal{G}$ have identical marginals. Then, the following are equivalent.*

1. $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$.
2. $Prob(\mathbf{X}_j \in K | \mathbf{X}_i = s) \geqslant Prob(\mathbf{Y}_j \in K | \mathbf{Y}_i = s) \quad \forall i, j, s, K \ni s$.
3. *For all $s \in \mathcal{S}$ and $K \subseteq \mathcal{S}$ such that $s \in K$, and for all i,*

$$\mathbb{E}\left[\mathbf{I}(K) \middle| \mathbf{X}_i = s\right] \geqslant \mathbb{E}\left[\mathbf{I}(K) \middle| \mathbf{Y}_i = s\right].$$

2.2. Strong Concentration along a Diagonal

The formulation of weak CAD in terms of $\mathbf{I}(K)$ suggests that one can construct other orders of similarity by ranking other properties of $\mathbf{H}_{\mathcal{F}}^{s,K}$, the distribution of $\mathbf{I}(K)$. We define an order called strong CAD below, which orders the conditional distributions $\mathbf{H}_{\mathcal{F}}^{s,K}$ not in terms of their expectations as in wCAD, but in terms of stochastic dominance.

Definition 2 (Strong Concentration along a Diagonal): *We say that $\mathcal{F}$ has a "strongly higher concentration along a diagonal" than $\mathcal{G}$, or $\mathcal{F}$* ***is sCAD higher than*** *$\mathcal{G}$, denoted by $\mathcal{F} \geqslant_{sCAD} \mathcal{G}$, if,*

1. *$\mathcal{F}$ and $\mathcal{G}$ have the same marginal distributions. And,*
2. *$\mathbf{H}_{\mathcal{F}}^{s,K} \geqslant_{st} \mathbf{H}_{\mathcal{G}}^{s,K}$ for all $s, K \ni s$, where $\geqslant_{st}$ denote first-order stochastic dominance.*

An increase in similarity, as measured by weak (or strong) CAD, requires that any two agents see *exactly the same* signal with a higher probability. This may be too strong in some contexts, making the orders quite incomplete. Suppose that the signals are ordered. Indeed, two information structures $\mathcal{F}$ and $\mathcal{G}$ are not comparable even if the probability of events in which $\mathcal{F}$ and $\mathcal{G}$ are *very close* in value is higher under $\mathcal{F}$, and the probability of events in which $\mathcal{F}$ and $\mathcal{G}$ are *very* *different* in value is lower under $\mathcal{F}$.

2.3. Concentration along a Diagonal: Contour Sets

Finally, we introduce a weaker (less incomplete) order of similarity called *Contour-set CAD* that uses the ordering of signals. Intuitively, increased similarity according to the contour-set CAD order, no longer means that there is a higher chance of getting exactly the same signal. Rather, $\mathcal{F}$ is more similar than $\mathcal{G}$ if the signals are close to each other in value with a higher probability under $\mathcal{F}$ than under $\mathcal{G}$.

Formally, assume that the set of signals $\mathcal{S}$ is an ordered set with $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$. Given any $\widehat{s} \in \mathcal{S}$, we define upper and lower contour sets of $\widehat{s}$.

$$\widehat{s}^{\uparrow} := \{y \in \mathcal{S} : y \geqslant \widehat{s}\}.$$

$$\widehat{s}^{\downarrow} := \{y \in \mathcal{S} : y \leqslant \widehat{s}\}.$$

Definition 3 (Concentration along a Diagonal: Contour Sets): *We say that $\mathcal{F}$ has a "higher concentration on the contour sets along a diagonal" than $\mathcal{G}$, or $\mathcal{F}$ **is cCAD higher than** $\mathcal{G}$, denoted by $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$, if,*

1. *$\mathcal{F}$ and $\mathcal{G}$ have the same marginal distributions. And,*
2. *For all $s \in \mathcal{S}$,*
   (a) *$\mathcal{F}_s(\widehat{s}^{\uparrow}) \geqslant \mathcal{G}(\widehat{s}^{\uparrow})$ for all $\widehat{s} \leqslant s$, and*
   (b) *$\mathcal{F}_s(\widehat{s}^{\downarrow}) \geqslant \mathcal{G}(\widehat{s}^{\downarrow})$ for all $\widehat{s} \geqslant s$*

The contour set CAD order can be stated in terms of intervals. It turns out that $\mathcal{F}$ has a higher concentration on the contour sets along a diagonal than $\mathcal{G}$, if and only if conditional on being of type s, an agent i assigns higher probability to any other agent j having type in *any interval* of types that includes type s.

Definition 4 (Concentration along a Diagonal: Intervals): *We say that $\mathcal{F}$ has a "higher concentration on the intervals along a diagonal" than $\mathcal{G}$, or $\mathcal{F}$ **is iCAD higher than** $\mathcal{G}$, denoted by $\mathcal{F} \succcurlyeq_{iCAD} \mathcal{G}$, if,*

1. *$\mathcal{F}$ and $\mathcal{G}$ have the same marginal distributions. And,*
2. *For all $s \in \mathcal{S}$, $\mathcal{F}_s(K) \geqslant \mathcal{G}_s(K)$ for all the intervals $K \ni s$.*

We state the equivalence of $\succcurlyeq_{cCAD}$ and $\succcurlyeq_{iCAD}$ formally in Proposition 1.

2.4. Comparing the notions

The next result characterizes the relationship between these three orders. The relationship is as a reader might expect.

Proposition 1: *Let $\mathbf{X}, \mathbf{Y}$ be two $\mathcal{S}^N$ valued random variables, where $\mathcal{S}$ is an ordered set.*

1. *If $N = 2$, then $\succcurlyeq_{sCAD} \Longleftrightarrow \succcurlyeq_{wCAD} \Longrightarrow \succcurlyeq_{cCAD} \Longleftrightarrow \succcurlyeq_{iCAD}$.*

2. If $N > 2$, then $\succcurlyeq_{sCAD} \implies \succcurlyeq_{wCAD} \implies \succcurlyeq_{cCAD} \iff \succcurlyeq_{iCAD}$.

Proof of Proposition 1. For all N, it is trivial that $\succcurlyeq_{sCAD} \implies \succcurlyeq_{wCAD}$. Moreover, $\succcurlyeq_{wCAD} \implies \succcurlyeq_{cCAD}$ follows by invoking Lemma 1 and letting K be any upper- or lower-contour sets. Since contour sets are intervals $\succcurlyeq_{iCAD} \implies \succcurlyeq_{cCAD}$ is also trivially true.

Next, we prove $\succcurlyeq_{cCAD} \implies \succcurlyeq_{iCAD}$. Suppose that $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$ but $\mathcal{F} \not\succcurlyeq_{iCAD} \mathcal{G}$. Therefore, $\exists\ s \in \mathcal{S}$ and an interval $K \ni s$ such that $\mathcal{F}_s(K) < \mathcal{G}_s(K)$. Let $\underline{s}, \bar{s}$ be the min and max elements of K respectively. We have,

$$\begin{aligned} 1 =& \mathcal{F}_s(\{s' : s' < \underline{s}\}) + \mathcal{F}_s(K) + \mathcal{F}_s(\{s' : s' > \bar{s}\}) \\ =& \mathcal{G}_s(\{s' : s' < \underline{s}\}) + \mathcal{G}_s(K) + \mathcal{G}_s(\{s' : s' > \bar{s}\}) \end{aligned}$$

Therefore, at least one of the following holds:

(*i*) $\mathcal{F}_s(\{s' : s' < \underline{s}\}) > \mathcal{G}_s(\{s' : s' < \underline{s}\})$.

(*ii*) $\mathcal{F}_s(\{s' : s' > \bar{s}\}) > \mathcal{G}_s(\{s' : s' > \bar{s}\})$

If (*i*) holds, then $\mathcal{F}_s(\underline{s}^{\uparrow}) < \mathcal{G}_s(\underline{s}^{\uparrow})$. If (*ii*) holds, then $\mathcal{F}_s(\bar{s}^{\downarrow}) < \mathcal{G}_s(\bar{s}^{\downarrow})$. In either case, this contradictions $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$.

To see that $\succcurlyeq_{cCAD}$ does not imply $\succcurlyeq_{wCAD}$, consider Figure 2. We start with a random variable $\mathbf{X}$ with a support $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$. We construct a random variable $\mathbf{Y}$ by increasing the mass on realizations $(1, 1), (2, 3), (3, 2)$, and $(4, 4)$ by $\alpha > 0$ each, and reducing the mass on realizations $(1, 3), (2, 4), (3, 1)$, and$(4, 2)$ by α. The marginal distributions of $\mathbf{X}$ and $\mathbf{Y}$ coincide. Moreover, $\mathbf{X} \succcurlyeq_{cCAD} \mathbf{Y}$ but $\mathbf{X} \not\succcurlyeq_{wCAD} \mathbf{Y}$.

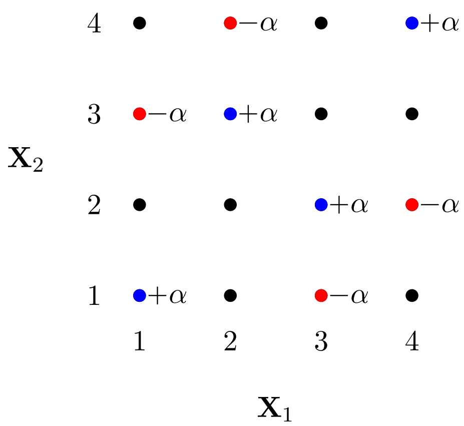


Figure 2: $\mathbf{X} \succcurlyeq_{cCAD} \mathbf{Y}$ but $\mathbf{X} \not\succcurlyeq_{wCAD} \mathbf{Y}$.

Finally, we show that $\succcurlyeq_{wCAD}$ does not imply $\succcurlyeq_{sCAD}$ except for $N = 2$. Suppose $N = 2$. To show that $\succcurlyeq_{wCAD} \implies \succcurlyeq_{sCAD}$, notice that $\mathbf{I}(\cdot)$ is a $\{0, 1\}$-valued random variable when $N = 2$. Therefore, for any $K \subseteq \mathcal{S}$,

$$\mathbb{E}_{\mathbf{H}^{s,K}_{\mathcal{F}}}(\mathbf{I}(K)) > \mathbb{E}_{\mathbf{H}^{s,K}_{\mathcal{G}}}(\mathbf{I}(K)) \iff \mathbf{H}^{s,K}_{\mathcal{F}}(\{\mathbf{I}(K) = 1\}) > \mathbf{H}^{s,K}_{\mathcal{F}}(\{\mathbf{I}(K) = 1)\}).$$

We show that $\succcurlyeq_{wCAD}$ does not imply $\succcurlyeq_{sCAD}$ for $N > 2$ also by construction. Let $\mathcal{S} = \{1, 2, 3\}$ and $\mathbf{X}$ be some $\mathcal{S}^3$-valued random variable. Construct $\mathbf{Y}$ from $\mathbf{X}$ by doing the following:

1. Increase the mass by $\alpha > 0$ on the events where any pair of $\mathbf{X}_i, \mathbf{X}_j$ are equal, but not all 3 are equal.
2. Reduce the mass by $\beta = 3\alpha$ on the events where all three realizations are unequal.

It is easy to check that the marginal distributions are the same, and $\mathbf{X} \succcurlyeq_{wCAD} \mathbf{Y}$, but $\mathbf{X} \not\succcurlyeq_{sCAD} \mathbf{Y}$. □

In the appendix A.9, we discuss the relationship between the CAD orders and other well-known orders such as supermodular order or concordance order. For $N = 2$, our orders imply the supermodular and positive quadrant dependence order, and for $N > 2$ the notions are not nested.

3. CAD Similarity and Coordination Equivalence

Why are the CAD orders appropriate for studying the effect of information similarity on strategic behavior? Our starting premise is that any economically meaningful similarity ranking of information structures in strategic settings should have at least the following feature: If agents have common interests and are engaged in a pure coordination game in the presence of some incomplete information (about the state of the world or others' types), then giving these players information that is more similar (greater in the chosen order) must surely aid coordination.

Arguably, there are many ways of formalizing what it means to aid coordination. We consider binary-action (0 or 1) coordination games under incomplete information and view aiding coordination as expanding the set of equilibria. To see why this is a natural criterion, consider the case of perfectly homogeneous signals—that is, all agents always see the same signal. Then, in a coordination game, for any given signal, all players playing action 1 or all players playing action 0 both constitute equilibria (assuming the actions are undominated). However, if the signals are not perfectly homogeneous, for some signal realizations, an agent may not have the incentive to play a certain action even if all other agents play the same action at that signal. In general, when information becomes more similar, in a binary-action coordination game, the incentive to play either action is relaxed if others play the same action, resulting in a larger set of equilibria. As the example in the introduction highlights, somewhat surprisingly, increasing similarity in terms of increases in existing orders like supermodular order or positive quadrant dependence do not aid coordination even in simple coordination games with common interests.[9]

[9]Bergemann and Morris (2016) also use equilibrium set inclusion to define a partial order of informativeness in games of incomplete information. The authors propose that more information

Our main results, presented in this section, show that concentration-along-the-diagonal is in fact precisely the correct notion for comparing information similarity in strategic settings. In particular, we establish that in canonical classes of binary-action coordination games, increasing information similarity across agents in the sense of increasing CAD is *equivalent* to expanding the set of equilibria.[10]

3.1. Binary-action private-value coordination games

Consider a setting in which N players, indexed by i or $j \neq i$, simultaneously and independently choose whether to take an action ($a_i = 1$) or not ($a_i = 0$). Each player i's payoff depends on the aggregate action by others

$$\mathbf{A_{-i}} = \sum_{j \neq i} a_j.$$

Using the same notation as in Section 2, we let $\mathcal{S}$ be a fixed, finite set from which player types are drawn. Let $\vec{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N)$ be a $\mathcal{S}^N-$valued random variable distributed according to an exchangeable distribution $\mathcal{F}$. Suppose that $\vec{s} = (s_1, s_2, \ldots s_N) \in \mathcal{S}^N$ is the realized type profile. Player i knows her own type s_i but does not know a different player j's type s_j. As before, for any two agents i and $j \neq i$, and any $K \subset \mathcal{S}$ define $\mathcal{F}_s(\cdot) \in \Delta(\mathcal{S})$ by

$$\mathcal{F}_s(K) := Prob(\mathbf{X}_j \in K | \mathbf{X}_i = s).$$

In particular, $\mathcal{F}_s(s')$ means $\mathcal{F}_s(\{\mathbf{X}_j = s'\})$.

A player i with type $\mathbf{X}_i = s_i$ gets a payoff of $u(a_i = 1, \mathbf{A_{-i}}, s_i)$ if she acts and $u(a_i = 0, \mathbf{A_{-i}}, s_i)$ if she does not act. The net payoff from taking action $a = 1$ is

$$d(\mathbf{A_{-i}}, s_i) := u(a_i = 1, \mathbf{A_{-i}}, s_i) - u(a_i = 0, \mathbf{A_{-i}}, s_i) = \alpha(s_i) + \beta(s_i)h(\mathbf{A_{-i}}) \quad (1)$$

where $h(\cdot)$ is increasing and $\beta(\cdot)$ is non-negative. Since $\beta(\cdot) \geqslant 0$ for all $s \in \mathcal{S}$, the game Γ exhibits strategic complementarity. Henceforth, we refer to such symmetric binary-action games of strategic complementarity simply as *private-value coordination games*.

Definition 5 ((Affine) Private-value Coordination games): *A game Γ a called a* ***private-value coordination game*** *when the payoff difference can be described by Equation (1), with increasing $h(\cdot)$ and non-negative $\beta(\cdot)$. The coordination game Γ is said to be* ***affine*** *if $h(\cdot)$ is affine.*

Assumption 1: ***Payoff-relevant signals:*** *For every $i \in \mathbf{N}$ and $s_i, s_i' \in \mathcal{S}$ such*

shrinks the set of Bayes Correlated Equilibria because if agents have more information, then a smaller set of outcomes is incentive-compatible.

[10]This paper studies the effect of increased similarity of information in strategic settings. A related question is how increased similarity of information across agents may affect their actions in independent *decision problems*. The key difference is that conditional beliefs are not as relevant in decision problems. The interested reader can contact the authors for results in this setting.

that $s_i \neq s_i'$, *there is* A_{-i} *such that* $d(A_{-i}, s_i) \neq d(A_{-i}, s_i')$.

This assumption simply precludes the possibility of artificially creating finer signal spaces from the original signal space to render the notion of changing similarity vacuous.[11]

Technology adoption by firms is an example of such a game. To fix ideas, consider N firms choosing whether or not to invest in a new technology. A firm's type s_i captures the value of the new technology to her. Each firm's payoff from investment depends also on the aggregate investment. The non-negative $\beta(.)$ captures the network externality (see Farrell and Saloner, 1986; Katz and Shapiro, 1985), where each firm benefits when more firms invest in the new technology. Increasing $\alpha(s_i)$ means higher types derive more payoff from investing, whereas increasing $\beta(s_i)$ means higher types are more affected by aggregate investment. Other examples of private-value coordination games include retailers choosing whether to lease a real estate in a mall (see Sakovics and Steiner, 2012) or consumers choosing platforms (see Farrell and Klemperer, 2007).

Equilibrium

We restrict attention to symmetric Bayes Nash Equilibrium in pure strategies (henceforth, equilibrium). Under distribution $\mathcal{F}$, a strategy profile where each player plays $\sigma : \mathcal{S} \to \{0, 1\}$ constitutes an equilibrium if

$$\sigma(s) = 1 \implies \mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{F}] \geqslant 0 \quad \text{(IC:P)}$$
$$\sigma(s) = 0 \implies \mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{F}] \leqslant 0 \quad \text{(IC:NP)}$$

Let $\mathcal{E}(\Gamma, \mathcal{F})$ be the set of equilibrium in the game Γ under distribution $\mathcal{F}$. The restriction to symmetric pure strategies implies that any strategy σ simply partitions the signal space into participation and non-participation sets — sets of signals after which agents take action and do not take action respectively.

$$P(\sigma) := \{s \in \mathcal{S} : \sigma(s) = 1\}, \; NP(\sigma) := \mathcal{S} \backslash P(\sigma).$$

3.2. (w,s)-CAD Similarity and Coordination Equivalence

Our first result, Theorem 1 shows that increasing similarity of information in the wCAD order is equivalent to expanding the set of equilibria in the class of games Γ when $h(\cdot)$ is affine.

THEOREM 1: *Let $\mathcal{F}$ and $\mathcal{G}$ be two joint distributions over $\mathcal{S}^N$. The following are equivalent.*

1. $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$.

[11] For instance, in Example 1, a new signal space can be created by "splitting" each signal into two signals by adding a payoff irrelevant feature, such as color. Then, a wCAD increase in the original signal space may not translate to a wCAD increase in the new signal space.

2. $\mathcal{E}(\Gamma, \mathcal{F}) \supseteq \mathcal{E}(\Gamma, \mathcal{G})$ *for all affine private-value coordination games* Γ*.*

Proof of Theorem 1. We first show (1) $\implies$ (2). Suppose that $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$. Let $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})$ for some game Γ with $\beta(s) \geqslant 0$ for all $s \in \mathcal{S}$. We show that σ remains an equilibrium under the more wCAD similar information structure $\mathcal{F}$, i.e., $\sigma \in \mathcal{E}(\Gamma, \mathcal{F})$.

Since σ constitutes an equilibrium under $\mathcal{G}$, (IC:P) must hold for any $s \in P(\sigma)$, and (IC:NP) must hold for any $s \in NP(\sigma)$. Fix an agent i. We compare the net payoffs from taking action $a = 1$ under $\mathcal{F}$ and $\mathcal{G}$.

$$\begin{aligned}&\mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{F}] - \mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{G}] \\ =&\beta(s)\left[\mathbb{E}[h(\mathbf{A_{-i}})|s, \sigma; \mathcal{F}] - \mathbb{E}[h(\mathbf{A_{-i}})|s, \sigma; \mathcal{G}]\right].\end{aligned}$$

Since $h(\cdot)$ is affine and increasing, $h(y) = ky + l$ for some $k > 0$. So we can rewrite the above as follows:

$$\begin{aligned}=&k\beta(s)\left(\mathbb{E}\left[\mathbf{A_{-i}}\middle|s; \mathcal{F}\right] - \mathbb{E}\left[\mathbf{A_{-i}}\middle|s; \mathcal{G}\right]\right) \\ =&k\beta(s)\left(\mathbb{E}\left[\sum_{j \neq i} \mathbb{1}_{\mathbf{X}_j \in P(\sigma)}\middle|s; \mathcal{F}\right] - \mathbb{E}\left[\sum_{j \neq i} \mathbb{1}_{\mathbf{X}_j \in P(\sigma)}\middle|s; \mathcal{G}\right]\right) \\ =&\beta(s)k\left[\mathbb{E}\left[\mathbf{I}(P(\sigma))\middle|s; \mathcal{F}\right] - \mathbb{E}\left[\mathbf{I}(P(\sigma))\middle|s; \mathcal{G}\right]\right],\end{aligned}$$

where, recall that $\mathbf{I}(P(\sigma))$ counts the number of players other than player i with signal in set $P(\sigma)$. By Lemma 1 and since $\beta(s) \geqslant 0 \forall s \in \mathcal{S}$, the above expression is non-negative for $s \in P(\sigma)$ and non-positive for $s \in NP(\sigma)$. Therefore, for strategy profile σ, we have

$$\begin{aligned}\mathbb{E}[d(\mathbf{A_{-i}}, s, \boldsymbol{\theta})|s, \sigma; \mathcal{F}] - \mathbb{E}[d(\mathbf{A_{-i}}, s, \boldsymbol{\theta})|s, \sigma; \mathcal{G}] &\geqslant 0 \quad \text{if } s \in P(\sigma) \\ &\leqslant 0 \quad \text{if } s \in NP(\sigma) \qquad (2)\end{aligned}$$

Therefore, (IC:P) continues to hold for all $s \in P(\sigma)$ under $\mathcal{F}$ and (IC:NP) holds for all $s \in NP(\sigma)$ under $\mathcal{F}$.

Next, we show (2) $\implies$ (1). Suppose that $\mathcal{E}(\Gamma, \mathcal{F}) \supseteq \mathcal{E}(\Gamma, \mathcal{G})$ for all Γ with $\beta(s) \geqslant 0$ for all $s \in \mathcal{S}$, but $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$. Using Lemma 1, we know that $\exists s^* \in \mathcal{S}$ and $K \ni s^*$ such that,

$$\mathcal{F}(\mathbf{X}_j \in K|\mathbf{X}_i = s^*) < \mathcal{G}(\mathbf{X}_j \in K|\mathbf{X}_i = s^*).$$

The proof approach will be to establish a contradiction by constructing a game Γ, and a strategy profile σ such that $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})$ but $\sigma \notin \mathcal{E}(\Gamma, \mathcal{F})$.

Define σ such that

$$P(\sigma) = K, \qquad NP(\sigma) = \mathcal{S} \backslash K.$$

For $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})$, we need the following:

$$\mathbb{E}\left[\alpha(s) + \beta(s)h(\mathbf{A_{-i}})\middle|s\right] \geqslant 0 \text{ if } s \in K$$
$$\leqslant 0 \text{ if } s \notin K.$$

So, we construct a game Γ with

$$\begin{aligned}
\alpha(s) &= -1 \\
h(A_{-i}) &= A_{-i} \\
\beta(s) & \begin{cases} = 0 & \text{if } s \notin K \\ = \dfrac{1}{\mathbb{E}\left[\mathbf{A_{-i}}\middle|s^*;\mathcal{G}\right]} & \text{if } s \in K \text{ and } s = s^* \\ \geqslant \dfrac{1}{\mathbb{E}\left[\mathbf{A_{-i}}\middle|s;\mathcal{G}\right]} & \text{if } s \in K \text{ and } s \neq s^*. \end{cases}
\end{aligned}$$

With the above choice of $\beta(\cdot)$, it is easy to see that under $\mathcal{G}$, (IC:P) is satisfied for all $s \in K$, and (IC:NP) is satisfied for all $s \notin K$. Therefore, $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})$. Finally, by Lemma 1, $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$ is equivalent to

$$\mathbb{E}\left[\mathbf{A_{-i}}\middle|s^*;\mathcal{G}\right] > \mathbb{E}\left[\mathbf{A_{-i}}\middle|s^*;\mathcal{F}\right].$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[d(\mathbf{A_{-i}}, s^*)\middle|s^*;\mathcal{F}\right] &= -1 + \beta(s^*)\mathbb{E}\left[\mathbf{A_{-i}}\middle|s^*;\mathcal{F}\right] \\
&= -1 + \frac{1}{\mathbb{E}\left[\mathbf{A_{-i}}\middle|s^*;\mathcal{G}\right]}\mathbb{E}[\mathbf{A_{-i}}|s^*;\mathcal{F}] < 0
\end{aligned}$$

Therefore, $\sigma \notin \mathcal{E}(\Gamma, \mathcal{F})$. Note that our constructed game Γ exhibits complementarity, that is, $\beta(.) \geqslant 0$, and it contradicts (2).

□

Our next result considers games Γ in which $h(\cdot)$ is not necessarily affine, but simply increasing. Analogous to Theorem 1, we can show that increasing similarity of information according to the strong CAD order is equivalent to expanding the set of equilibria in this class of games.

Theorem 2: *Let $\mathcal{F}$ and $\mathcal{G}$ be two joint distributions over $\mathcal{S}^N$. The following are equivalent.*

1. $\mathcal{F} \succcurlyeq_{sCAD} \mathcal{G}$.
2. $\mathcal{E}(\Gamma, \mathcal{F}) \supseteq \mathcal{E}(\Gamma, \mathcal{G})$ *for all private-value coordination games* Γ.

The proof is similar to that of Theorem 1 and is in the appendix.[12]

Remark 1 (Equilibrium Selection and Welfare)**:** *The above results show that in a coordination game, increased similarity of information in the CAD orders expands the set of equilibria. A relevant question is the welfare implication of this increased similarity. A social planner may want the agents to play one action rather than the other. Therefore, whether the social planner prefers more or less homogenization of information depends on whether the agents play the advantageous or the adversarial equilibrium. For instance, consider Example 1 and suppose a social planner prefers investment. If the social planner anticipates the agents will play the advantageous (maximal investment) equilibrium then she prefers greater homogenization of information.*[13] *Conversely, if the social planner anticipates the agents will play the adversarial (minimal investment) equilibrium then she prefers lesser homogenization of information.*[14]

3.3. Common-value Affine Coordination Games

Theorems 1 and 2 establish an equivalence between increases in similarity in CAD orders and the expansion of the set of *all* pure strategy equilibria. But in many economic applications, it is reasonable to consider some structure in the equilibrium strategies: For instance, in this section, we study such a class of coordination games commonly studied in the global games literature, in which it is typical to focus on equilibria in cutoff strategies— a player acts if and only if her type is high enough.[15] We establish an analogous equivalence between increasing similarity in the contour-CAD order and expanding the set of cutoff equilibria.

As in Section 3.1, we consider a setting in which N players simultaneously and independently choose whether to act ($a_i = 1$) or not ($a_i = 0$). But now, each player i's payoff depends on the aggregate action by others and an unknown but common underlying state $\boldsymbol{\theta} \in \Theta$, where Θ is a finite ordered set. The state is drawn from a common prior $\mu_0 \in \Delta(\Theta)$. We interpret an agent's type as the private

[12] Our main results make the case for using CAD to study the effect of information similarity on equilibrium behavior. The reader may wonder about the relationship between CAD notions and weaker solution concepts like rationalizability. In the online appendix Section B.2, we show, for a coordination game that if an action is rationalizable, then it continues to be so when the types become more similar in the sense of increasing wCAD (or sCAD).

[13] To see a numerical example, consider the marginal distribution of signal $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ in Example 1. The maximal investment equilibrium under an independent signal is the one where agents invest only when they see signal $\frac{3}{2}$. However, when they see perfectly homogeneous signals, the maximal investment equilibrium is one where agents invest when they see signals in $\{\frac{1}{2}, \frac{3}{2}\}$.

[14] To see a numerical example, consider the marginal distribution of signal $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$ in Example 1. The minimal investment equilibrium under independent signal is the one where agents invest when they see signal $\{\frac{1}{2}, \frac{3}{2}\}$. However, when they see perfectly homogeneous signals, the minimal investment equilibrium is one where agents invest only when they see signals in $\frac{3}{2}$.

[15] The maximal and minimal equilibria in these settings are in cutoff strategies.

signal she receives about the underlying state. Let $\mathcal{S}$ be a fixed, finite ordered set of signals, and the profile of signals $\vec{\mathbf{X}}$ be a $\mathcal{S}^N-$valued random variable. Conditional on $\boldsymbol{\theta} = \theta$, $\vec{\mathbf{X}}$ is distributed according to $\mathcal{F}^\theta$. We assume that $\mathcal{F}^\theta$ is exchangeable for all $\theta \in \Theta$. Therefore, $\mathcal{F}$ is also exchangeable.

A player i receives a signal $\mathbf{X}_i$. The marginal distribution of $\mathbf{X}_i$ conditional on state $\boldsymbol{\theta} = \theta$ is denoted by $\text{marg}\mathcal{F}^\theta(.) = \sum_{s_{-i}\in\mathcal{S}^{N-1}} \mathcal{F}^\theta(., s_{-i})$. Every signal realization $\mathbf{X}_i = s$ generates a posterior distribution over Θ obtained using Bayes rule. That is, for any $T \subseteq \Theta$,

$$\mu(s)(T) = \frac{\sum_{\theta\in T} \mu_0(\theta)\text{marg}\mathcal{F}^\theta(s)}{\sum_{\theta\in\Theta} \mu_0(\theta)\text{marg}\mathcal{F}^\theta(s)}$$

We assume that $\sum_{\theta\in\Theta} \mu_0(\theta)\text{marg}\mathcal{F}^\theta(s) > 0$ for all $s \in \mathcal{S}$. Conditional on $\boldsymbol{\theta} = \theta$, as before, for any i and $j \neq i$, and any $K \subset \mathcal{S}$, we define $\mathcal{F}^\theta_s(\cdot) \in \Delta(\mathcal{S})$ by

$$\mathcal{F}^\theta_s(K) := Prob(\mathbf{X}_j \in K|\mathbf{X}_i = s, \boldsymbol{\theta} = \theta).$$

In particular, $\mathcal{F}^\theta_s(s')$ means $\mathcal{F}^\theta_s(\{\mathbf{X}_j = s'\})$. Exchangeability of $\mathcal{F}^\theta$ implies that we need not index these conditional distributions with player identities.

The net payoff of each player takes the following form.

$$d(A_{-i}, \theta) = \alpha(\theta) + \beta(\theta)h(A_{-i}), \tag{3}$$

for some affine, increasing $h(\cdot)$ and $\beta(\cdot) \geqslant 0$. We make an assumption similar to Assumption 1.

Assumption 2: *[**Payoff-relevant state**] For every $i \in \mathbf{N}$ and $\theta \neq \theta'$, there is A_{-i} such that $d(A_{-i}, \theta) \neq d(A_{-i}, \theta')$.*

Notice that unlike in the games described in Section 3.1, players' payoffs do not depend on their idiosyncratic type. Rather, this is a "common-value" environment, where payoffs depend on a common unknown state.

Definition 6 (Common-value Affine Coordination games): *A game $\widetilde{\Gamma}$ a called a **common-value affine coordination game** when the payoff difference can be described by Equation (3), with increasing and affine $h(\cdot)$ and non-negative $\beta(\cdot)$.*

There are many applications with this payoff structure. A famous example is a currency attack game à la Morris and Shin (1998). Speculators simultaneously decide whether or not to attack a fixed exchange rate regime by selling the currency short. The payoff from not shorting is $u(a_i = 1, A_{-i}, \theta) = 0$ and the payoff from shorting is $u(a_i = 0, A_{-i}, \theta) = b(\theta)(1 - p(\theta, A_{-i})) - cp(\theta, A_{-i})$, where $p(\theta, A_{-i})$ is the probability that the currency will not be devalued, $b(\theta) > 0$ is the benefit when the currency is devalued, and c is the cost when it is not devalued. This gives $d(A_{-i}, \theta) = (b(\theta) + c)p(\theta, A_{-i}) - c$. Let $p(\theta, A_{-i}) = p_\theta\theta + p_A h(A_{-i})$. The currency is less likely to be devalued when the currency is stronger ($p_\theta > 0$) and fewer speculators short ($h(A_{-i})$ affine and increasing and $p_A > 0$). Then,

$$d(A_{-i}, \theta) = \underbrace{p_\theta(b(\theta) + c)\theta - c}_{\alpha(\theta)} + \underbrace{p_A(b(\theta) + c)}_{\beta(\theta)>0} \cdot h(A_{-i}).$$

Similar payoff structures arise in other games with strategic complementarity like debt rollover problems (e.g., Morris and Shin (2004)) or bank runs (e.g.,Goldstein and Pauzner (2005)) when the probability of the borrower or the bank surviving is $p(\theta, A_{-i}) = p_\theta \theta + p_A h(A_{-i})$.

Equilibrium

We restrict attention to symmetric Bayes Nash Equilibrium in pure strategies (henceforth, equilibrium). Under distribution $\mathcal{F}$, a strategy profile where each player plays $\sigma : \mathcal{S} \to \{0, 1\}$ constitutes an equilibrium if

$$\sigma(s) = 1 \implies \mathbb{E}[d(\mathbf{A_{-i}}, , \boldsymbol{\theta})|s, \sigma; \mathcal{F}] \geqslant 0$$
$$\sigma(s) = 0 \implies \mathbb{E}[d(\mathbf{A_{-i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{F}] \leqslant 0.$$

Cutoff Strategies

As is customary in many applications, we focus on equilibria in cut-off strategies.

Definition 7: *We say a strategy σ is a cutoff equilibrium if there is a cutoff $\tilde{s}$ such that $P(\sigma) = \{s \in \mathcal{S} : s \geqslant \tilde{s}\}$ and $NP(\sigma) = \mathcal{S} \backslash P(\sigma)$. Let $\overline{\mathcal{E}}(\widetilde{\Gamma}, \mathcal{F})$ denote the set of cutoff equilibria given any common-value affine coordination game $\widetilde{\Gamma}$ and information structure $\mathcal{F}$.*

3.4. Contour-CAD and Cutoff Equilibria

We show below that in this class of games, increasing similarity in the contour CAD order implies an increase in the set of cut-off equilibria, and the converse is also true under an additional regularity condition.

Definition 8 (Affine independence): *An information structure $\mathcal{F}$ is **affinely independent** if the set of posteriors $\mu(s)$ it generates is affinely independent. That is,*

$$\sum_{s \in \mathcal{S}} \lambda_s \mu(s) = 0 \text{ and } \sum_{s \in \mathcal{S}} \lambda_s = 0 \implies \lambda_s = 0 \forall s \in \mathcal{S}.$$

Theorem 3: *Let $\mathcal{F} = (\mathcal{F}^\theta)_{\theta \in \Theta}$ and $\mathcal{G} = (\mathcal{G}^\theta)_{\theta \in \Theta}$ be two distributions with identical marginal distributions for any $\theta \in \Theta$.*

1. *If $\mathcal{F}^\theta \geqslant_{cCAD} \mathcal{G}^\theta$ for all θ with $\mathcal{F}^\theta \neq \mathcal{G}^\theta$ then $\overline{\mathcal{E}}(\widetilde{\Gamma}, \mathcal{F}) \supseteq \overline{\mathcal{E}}(\widetilde{\Gamma}, \mathcal{G})$ for all common-value affine coordination games $\widetilde{\Gamma}$.*
2. *Suppose $\mathcal{G}$ (and hence $\mathcal{F}$) is affinely independent. Let $T \subseteq \Theta$ be such that*

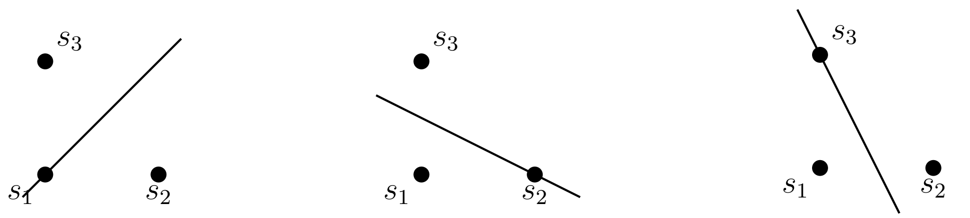
$\mathcal{F}^\theta \neq \mathcal{G}^\theta$ iff $\theta \in T$. Then $\overline{\mathcal{E}}(\widetilde{\Gamma}, \mathcal{F}) \supseteq \overline{\mathcal{E}}(\widetilde{\Gamma}, \mathcal{G})$ for all common-value affine coordination games $\widetilde{\Gamma} \implies \mathcal{F}^\theta \succcurlyeq_{cCAD} \mathcal{G}^\theta\ \forall \theta \in T$.

The proof of Theorem 3 is in the appendix. Below we explain why we impose the condition of affine independence of the posteriors $\mu(\cdot)$, that we did not need in proving Theorems 1 and 2. For part (2), we want to show that if $\mathcal{F}^\theta \not\succcurlyeq_{cCAD} \mathcal{G}^\theta$ for some $\theta \in \Theta$, then we can construct a game $\widetilde{\Gamma}$ such that $\overline{\mathcal{E}}(\widetilde{\Gamma}; \mathcal{G}) \nsubseteq \overline{\mathcal{E}}(\widetilde{\Gamma}; \mathcal{F})$. If $\mathcal{F}^\theta \not\succcurlyeq_{cCAD} \mathcal{G}^\theta$, then we have a signal s and a contour set $\widehat{s}^\uparrow$ such that $s \in \widehat{s}^\uparrow$ and $\mathcal{F}^\theta_s(\widehat{s}^\uparrow) < \mathcal{G}^\theta(\widehat{s}^\uparrow)$.[16] To prove the analogous parts of Theorems 1 and 2, we used the dependence of the payoff on the private signal $\beta(\cdot)$ to construct a game such that the desired equilibrium set inclusion fails. Now, $\beta(\cdot)$ depends only on the common θ, and so we cannot use the same approach.

To see how affine independence is used, consider three affinely independent points $\{s_1, s_2, s_3\}$. Partition this set into two disjoint sets, K_1, K_2. We can draw a supporting hyperplane passing through any point from $\{s_1, s_2, s_3\}$ that separates K_1 and K_2 (see Figure 3). This observation implies we can define a function $\alpha : \Theta \to \mathbb{R}$ such that,

$$\max_{s' < \widehat{s}} \mathbb{E}_{\mu(s')}[\alpha(\boldsymbol{\theta})] < \mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] < \min_{s' \in \widehat{s}^\uparrow, s' \neq s} \mathbb{E}_{\mu(s')}[\alpha(\boldsymbol{\theta})].$$

Using $\alpha(\cdot)$ defined above, and $\beta(\theta') = \mathbb{1}_{\theta' = \theta}$ we can construct a game $\widetilde{\Gamma}$ such that $\sigma(s') = \mathbb{1}_{s' \geqslant \widehat{s}}$ is an equilibrium in Γ under $\mathcal{G}$ but not under $\mathcal{F}$ leading to a desired contradiction. This argument also suggests that affine independence may not be the weakest requirement on the set of posteriors.



Supporting hyperplane through s_1 separating $\{s_1, s_2\}$ from $\{s_3\}$. | Supporting hyperplane through s_2 separating $\{s_2, s_3\}$ from $\{s_1\}$. | Supporting hyperplane through s_3 separating $\{s_1, s_3\}$ from $\{s_2\}$.

Figure 3: Illustration of supporting hyperplanes separating sets of points.

A commonly studied special case of this setting is the separable environment, in which β is a state-independent constant. In the appendix A.4, we show that Theorem 3 continues to hold in the separable environment. However, the difference is that we do not need the signals to be more similar in each state (i.e., $\mathcal{F}^\theta \succcurlyeq_{cCAD} \mathcal{G}^\theta$), but only in expectation.

[16] Alternatively, a contour set $\widehat{s}^\downarrow$ with an analogous implication that we choose to omit in this discussion for the sake of brevity.

4. Applications

Theorems 1 through 3 make the case for why CAD orders are well-suited to compare information similarity in Bayesian games, by showing the equivalence of increase in CAD-similarity and aiding coordination (expansion of the set of Bayes Nash Equilibrium) in canonical coordination games. In this section, we show how CAD orders can be used to derive economically meaningful predictions in settings much beyond the games considered so far.

4.1. Similarity and Congestion

Consider a binary-action game of strategic substitutability in which N players simultaneously choose between actions 0 or 1, and the payoff difference

$$d(\mathbf{A_{-i}}, s_i) := u(a_i = 1, \mathbf{A_{-i}}, s_i) - u(a_i = 0, \mathbf{A_{-i}}, s_i) = \alpha(s_i) + \beta(s_i)h(\mathbf{A_{-i}}) \quad (4)$$

decreases in the aggregate action, that is, $\beta(.) < 0$. Since $\beta(\cdot) < 0$ for all $s \in \mathcal{S}$, the game Γ exhibits strategic substitutability. Henceforth, we refer to such symmetric binary-action games of strategic substitutability simply as *congestion games*. In contrast to coordination games, a player in a congestion game has a *smaller* incentive to play action 1 if more of the other players play action 1.

Definition 9 ((Affine) Congestion Games): *A game $\hat{\Gamma}$ a called a* ***congestion game*** *when the payoff difference can be described by Equation (4), with increasing $h(\cdot)$ and negative $\beta(\cdot)$. The congestion game $\hat{\Gamma}$ is said to be* ***affine*** *if $h(\cdot)$ is affine.*

A classic example is an entry game in a market with capacity constraints. A firm suffers when more firms enter making the market more congested (see Duffy and Hopkins, 2005). The conflict game of Baliga and Sjöström (2012) is another interesting example in which a country wants to be aggressive whenever its opponent is peaceful and vice versa. A public good contribution game in which an agent's contribution to the public good matters less for successful public good provision when more agents contribute also exhibits strategic substitutability (see Harrison and Jara-Moroni, 2021).

We can establish equivalence results that are analogous to our main results: More CAD-similar information shrinks the set of symmetric pure strategy BNE in congestion games.

Proposition 2: *Let $\mathcal{F}$ and $\mathcal{G}$ be two joint distributions over $\mathcal{S}^N$.*

1. *$\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$ if and only if $\mathcal{E}(\Gamma, \mathcal{F}) \subseteq \mathcal{E}(\Gamma, \mathcal{G})$ for all affine congestion games $\hat{\Gamma}$.*
2. *$\mathcal{F} \succcurlyeq_{sCAD} \mathcal{G}$ if and only if $\mathcal{E}(\Gamma, \mathcal{F}) \subseteq \mathcal{E}(\Gamma, \mathcal{G})$ for all congestion games $\hat{\Gamma}$.*

4.2. Similarity and Collective Action

Recent public discourse and research in political science suggests that social media or access to the same information has enabled larger mass protests.[17] Protests are fundamentally collective action problems: Protesting is costly and a successful regime change requires a sufficient number of people to take this costly action, while the benefit of regime change accrues to all. So, while people want to coordinate to ensure a succcesful mass protest, they are also tempted to free-ride. Importantly, unlike the games we studied so far in this paper, collective action games do not exhibit strategic complementarity. In a different paper Basak et al. (2024), we show that the notion of weak CAD can still be used to characterize when increased information similarity helps or harms participation in a collective action game of incomplete information. The main insight is that more similar information about the fundamentals in the sense of increased CAD is a double-edged sword: It can help agents coordinate, but can also exacerbate free-riding. We show that more similar information facilitates (impedes) collective action when achieving regime change is sufficiently challenging (easy).

4.3. Similarity and Auctions

A classical question in auction theory (e.g., Milgrom and Weber (1982)) is how different auction formats compare in terms of revenue, when players' valuations are more interdependent. We can use the tools developed in this paper to answer a related question: Given an auction format, how does revenue change if player valuations become more similar? For example, below, we show that the revenue from a second-price auction unambiguously increases when players' valuations become more similar in the weak CAD order.

Consider a second-price auction for a single object with two bidders. Bidder valuations are drawn from a finite set $\mathcal{S}$. Let $\mathcal{F}$ and $\mathcal{G}$ be two different joint distributions over the bidder valuations. Let $R(\cdot)$ denote the expected revenue from the second-price auction given a joint distribution, when bidders report their valuations truthfully. Truthful reporting is weakly dominant strategy in this environment. The result below shows that when bidder valuations are more similar in the sense of an increase in the wCAD order, then the expected revenue in the second-price auction is higher. The proof is in the appendix.

Proposition 3: *If $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$, then $R(\mathcal{F}) \geqslant R(\mathcal{G})$.*

4.4. Coordination with continuum of states and signals

Throughout this paper, we consider coordination games with finite states and signals. However, the notion of cCAD-similarity can be easily used for a continuum of states and signals as well. A large literature has studied coordination problems

[17] See Manacorda and Tesei (2020), Qin et al. (2024) for example.

in financial and macroeconomic settings, in which agents are uncertain about a payoff-relevant state that is drawn from a continuum. It is typical in this literature to assume that agents have an improper uniform prior about the state, and receive private Gaussian signals before they choose their (binary) action. We can use our notion of contour-CAD also in this setting with a continuum of states and signals, to describe how increased similarity of information across agents affects the set of equilibria.

Consider a standard global game with two players. An unknown state $\theta \in \mathbb{R}$ is drawn from an improper prior $\theta \sim U(\mathbb{R})$. Each player $i = 1, 2$ receives a noisy private signal $\mathbf{X}_i = \boldsymbol{\theta} + \varepsilon_i$, where $(\varepsilon_1, \varepsilon_2)$ is independent of θ and drawn from a bi-variate normal distribution with correlation ρ.

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\right).$$

After observing the private signal, each agent i decides whether to take an action ($a_i = 1$) or not ($a_i = 0$). The payoff difference between taking action and not is

$$d(\theta, a_j) = \alpha(\theta) + \beta \cdot h(a_j), \tag{5}$$

where $h(.)$ is affine and $\beta > 0$.

The result below states that increasing correlation in this setting implies an increase in the contour-CAD order, which in turn implies an expansion in the set of equilibria.[18]

Proposition 4: *Under the information structure with an improper prior and bi-variate normal noise, as ρ increases, information becomes more similar in the cCAD sense. Accordingly, the set of equilibria expands for all games in which the payoff difference can be represented as (5).*

The proof is in the appendix. The logic of the proof is as follows. The payoff structure described by (5) is a special case of our payoff specification in Section 3.4, with β being independent of θ. So we can use Theorem 3 once we establish that increasing ρ implies a contour CAD increase. But this is subtle. In general, increasing the correlation of a joint distribution does not necessarily imply a contour CAD increase. But it turns out that when agents have an uninformative (improper) prior, increasing ρ also implies a cCAD increase.[19]

4.5. Games with non-exchangeable signal distributions

Our baseline setting features a lot of symmetry: Player identities are not payoff-relevant and the joint distribution over agents' signals is exchangeable. In Ap-

[18]Note that this increase in similarity is not conditional on θ.

[19]For a bi-variate normal distribution, Müller and Scarsini (2000) has shown that increasing correlation ρ is equivalent to an increase of interdependence according to the supermodular order.

pendix A.8, we relax these symmetry assumptions. We consider a class of binary-action coordination games in which payoffs depend on a weighted aggregate action by others (with identity-specific weights) and allow for non-exchangeable signal distributions. We define an analogous notion of weak CAD for non-exchangeable joint distributions, and derive a characterization analogous to Theorem 1.

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A. Appendix: Proofs

A.1. Proof of Lemma 1

Proof. First, we prove that 1. $\iff$ 2. Suppose that $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$ and for some K and $s \in K$ and i, j, we have $Prob(\mathbf{X}_j \in K|\mathbf{X}_i = s) < Prob(\mathbf{Y}_j \in K|\mathbf{Y}_i = s)$. Then, at least for some $s' \notin K$, $Prob(\mathbf{X}_j = s'|\mathbf{X}_i = s) < Prob(\mathbf{Y}_j = s'|\mathbf{Y}_i = s)$, contradicting $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$. For the converse, suppose that the inequality holds for all i, j, s and $K \ni s$, but $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$ does not hold. Then, for some $s' \neq s$, $Prob(\mathbf{X}_j = s'|\mathbf{X}_i = s) > Prob(\mathbf{Y}_j = s'|\mathbf{Y}_i = s)$. But then, $K = \mathcal{S}\backslash\{s'\}$ would have

$$Prob(\mathbf{X}_j \in K|\mathbf{X}_i = s) < Prob(\mathbf{Y}_j \in K|\mathbf{Y}_i = s),$$

a contradiction.

Next, we prove 1. $\iff$ 3. $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$

$$\begin{aligned}
&\iff Prob(\mathbf{X}_j \in K\Big|\mathbf{X}_i = s) \geqslant Prob(\mathbf{Y}_j \in K\Big|\mathbf{Y}_i = s) \quad \forall s, K \ni s,\\
&\iff \mathbb{E}\left[\mathbb{1}_{\mathbf{X}_j \in K}\Big|\mathbf{X}_i = s\right] \geqslant \mathbb{E}\left[\mathbb{1}_{\mathbf{Y}_j \in K}\Big|\mathbf{Y}_i = s\right] \quad \forall s, K \ni s.\\
&\iff (N-1)\mathbb{E}\left[\mathbb{1}_{\mathbf{X}_j \in K}\Big|\mathbf{X}_i = s\right] \geqslant (N-1)\mathbb{E}\left[\mathbb{1}_{\mathbf{Y}_j \in K}\Big|\mathbf{Y}_i = s\right] \quad \forall s, K \ni s.\\
&\iff \mathbb{E}\left[\sum_{j \neq i}\mathbb{1}_{\mathbf{X}_j \in K}\Big|\mathbf{X}_i = s\right] \geqslant \mathbb{E}\left[\sum_{j \neq i}\mathbb{1}_{\mathbf{Y}_j \in K}\Big|\mathbf{Y}_i = s\right] \quad \forall s, K \ni s.
\end{aligned}$$

□

A.2. Proof of Theorem 2

Proof. The proof approach is identical to that of Theorem 1. First, we establish (1) $\implies$ (2). Since $\mathcal{F} \succcurlyeq_{sCAD} \mathcal{G}$ for all increasing $h(\cdot)$,

$$\mathbb{E}[h(\mathbf{A_{-i}})|s;\mathcal{F}] \geqslant \mathbb{E}[h(\mathbf{A_{-i}})|s;\mathcal{G}] \quad \text{for all } \in T, s \in \mathcal{S}.$$

Given any σ, we can replicating the same argument, to obtain:

If $\beta(s) \geqslant 0 \quad \forall s \in \mathcal{S}$,

$$\mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{F}] - \mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{G}] \quad \begin{aligned} &\geqslant 0 \quad \text{if } s \in P(\sigma)\\ &\leqslant 0 \quad \text{if } s \in NP(\sigma)\end{aligned}$$

Therefore, for all Γ, if $\beta(s) \geqslant 0$ for all $s \in \mathcal{S}$, then $\sigma \in \mathcal{E}(\Gamma, \mathcal{G}) \implies \sigma \in \mathcal{E}(\Gamma, \mathcal{F})$.

Next, we establish (2) $\implies$ (1). We only sketch the argument for (2) $\implies$ (1), as the logic is very similar to that in Theorem 1. Suppose that $\mathcal{E}(\Gamma, \mathcal{F}) \supseteq \mathcal{E}(\Gamma, \mathcal{G})$

for all Γ with $\beta(s) \geqslant 0$ for all $s \in \mathcal{S}$, but $\mathcal{F} \not\succcurlyeq_{sCAD} \mathcal{G}$. Therefore, $\exists s^* \in \mathcal{S}$ and $K \ni s^*$ such that, for some $m < N-1$,

$$\mathcal{F}(\{\mathbf{I}(K) \geqslant m\}|\mathbf{X}_i = s^*) < \mathcal{G}(\{\mathbf{I}(K) \geqslant m\}|\mathbf{Y}_i = s^*).$$

We will construct a game Γ that exhibits strategic complementarity such that $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})$ but $\sigma \notin \mathcal{E}(\Gamma, \mathcal{F})$. Using the same idea as before, let σ be a strategy such that

$$P(\sigma) = K \qquad NP(\sigma) = \mathcal{S}\backslash K.$$

We specify a game Γ for which σ constitutes an equilibrium under $\mathcal{G}$.

$$\begin{aligned} h(A_{-i}) &= \mathbb{1}_{A_{-i}\geqslant m} \\ \alpha(\theta) &= -1 \\ \beta(\theta, s) &\begin{cases} = 0 & \text{if } s \notin K \\ = \dfrac{1}{\mathbb{E}\left[\mathbb{1}_{\mathbf{A_{-i}}\geqslant m}\middle| s^*; \mathcal{G}\right]} & \text{if } s \in K, s = s^* \\ \geqslant \dfrac{1}{\mathbb{E}\left[\mathbb{1}_{\mathbf{A_{-i}}\geqslant m}\middle| s; \mathcal{G}\right]} & \text{if } s \in K, s \neq s^* \end{cases} \end{aligned}$$

It is straightforward to verify that $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})\backslash\mathcal{E}(\Gamma, \mathcal{F})$, a contradiction. □

A.3. Proof of Theorem 3

We first establish a useful property of affine independence.

Lemma 2: *Let $x \in \mathbb{R}^n$ and $K, L, \subset \mathbb{R}^n$ be two finite, disjoint sets such that, $x \notin K \bigcup L$ and $K \bigcup L \bigcup \{x\}$ is affinely independent. Then, $\exists \vec{\lambda} \in \mathbb{R}^n$ such that*

$$\max_{k \in K} \lambda' k < \lambda' x < \min_{l \in L} \lambda' l.$$

Proof. Since $K \bigcup L \bigcup \{x\}$ is affinely independent, $\{k - x : k \in K\} \bigcup \{l - x : l \in L\}$ is linearly independent. Complete this set to a basis, $\{x_1, \ldots, x_n\}$ and define a linear map as follows:

$$T(x_i) := \begin{cases} -1 & \text{if } x_i = k - x \text{ for some } k \in K \\ 1 & \text{if } x_i = l - x \text{ for some } l \in L \\ 0 & \text{if } x_i \in T\backslash \left\{\{k - x : k \in K\} \bigcup \{l - x : l \in L\}\right\}. \end{cases}$$

$T(y)$ for any y is defined using the above basis vectors. By definition, $T(k - x) = T(k) - T(x) = -1 \implies T(k) < T(x)$ for all $k \in K$ and $T(l - x) = T(l) - T(x) = 1 \implies T(l) > T(x)$ for all $l \in L$. Finally, by the Riesz representation theorem, $\exists \lambda \in \mathbb{R}^n$ such that $T(y) = \lambda' y$ for all $y \in \mathbb{R}^n$. □

Proof of Theorem 3. To prove (1), define $T \subseteq \Theta$ with $\mathcal{F} = \mathcal{G}^\theta$ for all $\theta \in \Theta \backslash T$, and suppose that $\mathcal{F}^\theta \succcurlyeq_{cCAD} \mathcal{G}^\theta$ for all $\theta \in T$. Consider a common-value affine coordination game Γ. Affine, increasing $h(\cdot)$ means $h(x) = kx + l$ for some $k \geqslant 0$. Suppose that σ is a cutoff equilibrium under $\mathcal{G}$ with a cutoff $\tilde{s}$. Therefore,

$$\mathbb{E}\left[d(\mathbf{A_{-i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{G}\right] \geqslant 0 \quad \forall s \geqslant \tilde{s}$$
$$\mathbb{E}\left[d(\mathbf{A_{-i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{G}\right] \leqslant 0 \quad \forall s < \tilde{s}$$

For any s,

$$\begin{aligned}
&\mathbb{E}\left[d(\mathbf{A_{-i}}, \boldsymbol{\theta})|s; \sigma, \mathcal{F}\right] - \mathbb{E}\left[d(\mathbf{A_{-i}}, \boldsymbol{\theta})|s; \sigma, \mathcal{G}\right] \\
&= \mathbb{E}\left[\alpha(\boldsymbol{\theta}) + \beta(\boldsymbol{\theta})h(\mathbf{A_{-i}})|s; \sigma, \mathcal{F}\right] - \left[\alpha(\boldsymbol{\theta}) + \beta(\boldsymbol{\theta})h(\mathbf{A_{-i}})|s; \sigma, \mathcal{G}\right] \\
&= \mathbb{E}\left[\beta(\boldsymbol{\theta})k\mathbb{E}\left[\mathbf{A_{-i}}|\boldsymbol{\theta}, \mathcal{F}^{\boldsymbol{\theta}}\right]|s; \sigma\right] - \mathbb{E}\left[\beta(\boldsymbol{\theta})k\mathbb{E}\left[\mathbf{A_{-i}}|\boldsymbol{\theta}, \mathcal{G}^{\boldsymbol{\theta}}\right]|s; \sigma\right] \\
&= \mathbb{E}\left[\beta(\boldsymbol{\theta})k[\mathcal{F}_s^{\boldsymbol{\theta}}(\tilde{s}^\uparrow) - \mathcal{G}_s^{\boldsymbol{\theta}}(\tilde{s}^\uparrow)]|s; \sigma\right]
\end{aligned}$$

Since $\mathcal{F}^\theta \succcurlyeq_{cCAD} \mathcal{G}^\theta$ for all $\theta \in T$ and $\mathcal{F}^\theta = \mathcal{G}^\theta$ for all $\theta \notin T$, the above expression is non-negative if $s \geqslant \tilde{s}$ and non-positive otherwise. This implies that if σ was an equilibrium under $\mathcal{G}$, then it remains an equilibrium under $\mathcal{F}$, because (IC:P) continues to hold for all $s \geqslant \tilde{s}$, and (IC:NP) continues to hold for all $s < \tilde{s}$. Thus, $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{F})$, thus proving (1).

To prove (2), suppose that $\overline{\mathcal{E}}(\Gamma, \mathcal{F}) \supseteq \overline{\mathcal{E}}(\Gamma, \mathcal{G})$ for all common-value affine coordination games Γ. Suppose, $\mathcal{F}^{\theta*} \not\succcurlyeq_{cCAD} \mathcal{G}^{\theta*}$ for some $\theta^* \in T$. Then there exists $\tilde{s}, \hat{s}$ such that at least one of the following holds:

1. $\hat{s} \leqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}^{\theta*}(\hat{s}^\uparrow) < \mathcal{G}_{\tilde{s}}^{\theta*}(\hat{s}^\uparrow)$, or
2. $\hat{s} \geqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}^{\theta*}(\hat{s}^\downarrow) < \mathcal{G}_{\tilde{s}}^{\theta*}(\hat{s}^\downarrow)$.

Case 1: Suppose $\hat{s} \leqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}^{\theta*}(\hat{s}^\uparrow) < \mathcal{G}_{\tilde{s}}^{\theta*}(\hat{s}^\uparrow)$

We will derive a contradiction by constructing a common-value affine coordination game with payoff functions $d(A_{-i}, \theta) = \alpha(\theta) + \beta(\theta)h(A_{-i})$, and a strategy profile σ with

$$P(\sigma) = \hat{s}^\uparrow \qquad \text{and} \qquad NP(\sigma) = \mathcal{S} \backslash \hat{s}^\uparrow, \tag{6}$$

such that $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{G})$, but $\sigma \notin \overline{\mathcal{E}}(\Gamma, \mathcal{F})$.

Define

$$\begin{aligned}
A :=& \{\mu(s) : s \in \mathcal{S}, s < \hat{s}\} \\
\text{and } B :=& \{\mu(s) : s \in \mathcal{S}, s \geqslant \hat{s}, s \neq \tilde{s}\}.
\end{aligned}$$

Now $A \bigcup \{\mu(\tilde{s})\} \bigcup B$ is an affinely independent set. By Lemma 2, there exists $\tilde{\alpha} \in \mathbb{R}^N$ such that

$$\max_{\mu(s) \in A} \tilde{\alpha}'\mu(s) < \tilde{\alpha}'\mu(\tilde{s}) < \min_{\mu(s) \in B} \tilde{\alpha}'\mu(s).$$

Notice that $\widetilde{\alpha}'\mu(\cdot) = \mathbb{E}_{\mu(\cdot)}[\widetilde{\alpha}(\boldsymbol{\theta})]$. Therefore, $\exists\widetilde{\alpha} : \Theta \to \mathbb{R}$ such that,

$$\max_{\mu(s)\in A} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})] < \mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta})] < \min_{\mu(s)\in B} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})].$$

Define, for any s,

$$l(s) := \mu(s)(\theta^*)\mathcal{G}_s^{\theta^*}(\widehat{s}^{\uparrow}).$$

Define, for any $k \in \mathbb{R}$,

$$\Delta_1(k) := \left[\mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(\tilde{s})\right] - \left[\max_{\mu(s)\in A} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(s)\right],$$

$$\text{and } \Delta_2(k) := \left[\min_{\mu(s)\in B} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(s)\right] - \left[\mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(\tilde{s})\right]$$

Notice that $(\Delta_1(0), \Delta_2(0)) > (0, 0)$. By continuity in k, $\exists k > 0$ such that $(\Delta_1(k), \Delta_2(k)) > (0, 0)$. Fix any such k, and choose a scalar a so that

$$\mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta}) + a] + kl(\tilde{s}) = 0.$$

Define

$$\alpha(\cdot) := \widetilde{\alpha}(\cdot) + a,$$

and let

$$\beta(\theta) = \frac{k}{(N-1)} \mathbb{1}_{\theta=\theta^*}.$$

Consider a common-value affine coordination game Γ with $h(x) = x$, (α, β) defined above, and the strategy profile in (6) above. Note that we have

$$\max_{\mu(s)\in A} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + kl(s)\right\} < \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + kl(\tilde{s}) = 0 < \min_{\mu(s)\in B} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + kl(s)\right\}$$

Further, for any s,

$$\mathbb{E}_{\mu(s)}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_s^{\boldsymbol{\theta}}(\widehat{s}^{\uparrow})] = kl(s).$$

Therefore the following conditions are satisfied.

$$\begin{aligned} \max_{\mu(s)\in A} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(s)}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_s^{\boldsymbol{\theta}}(\widehat{s}^{\uparrow})]\right\} &<0 \\ \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(\tilde{s})}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_{\tilde{s}}^{\boldsymbol{\theta}}(\widehat{s}^{\uparrow})] &=0 \\ \min_{\mu(s)\in B} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(s)}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_s^{\boldsymbol{\theta}}(\widehat{s}^{\uparrow})]\right\} &>0. \end{aligned} \tag{7}$$

Equations (7) simply imply that the strategy $P(\sigma) = \widehat{s}^{\uparrow}$ and $NP(\sigma) = \mathcal{S}\backslash\widehat{s}^{\uparrow}$ constitutes an equilibrium under $\mathcal{G}$, i.e., $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{G})$. In particular, (IC:P) holds with an equality for $\tilde{s}$, and holds strictly for any $s > \widehat{s}$ such that $s \neq \tilde{s}$. Finally, notice that, since $\mathcal{F}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\uparrow}) < \mathcal{G}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\uparrow})$,

$$\begin{aligned}\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(\tilde{s})}[\beta(\boldsymbol{\theta})\mathcal{F}_{\tilde{s}}^{\boldsymbol{\theta}}(\widehat{s}^{\uparrow})] &= \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + k\mu(\tilde{s})(\theta^*)\mathcal{F}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\uparrow}) \\ &< \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + kl(\tilde{s}) = 0.\end{aligned}$$

Therefore, $\sigma \notin \overline{\mathcal{E}}(\Gamma, \mathcal{F})$, a contradiction. Thus, case 1 cannot hold.

Case 2: $\widehat{s} \geqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\downarrow}) < \mathcal{G}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\downarrow})$:

The proof is nearly identical to the previous case. Define

$$\begin{aligned}A :=& \{\mu(s) : s \in \mathcal{S}, s \leqslant \widehat{s}, , s \neq \tilde{s}\} \\ \text{and } B :=& \{\mu(s) : s \in \mathcal{S}, s > \widehat{s}\}.\end{aligned}$$

Notice that $\mathcal{F}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\downarrow}) < \mathcal{G}_{\tilde{s}}^{\theta^*}(\widehat{s}^{\downarrow}) \iff \mathcal{F}_{\tilde{s}}^{\theta^*}(B) > \mathcal{G}_{\tilde{s}}^{\theta^*}(B)$ since $B = \mathcal{S}\backslash\{\widehat{s}^{\downarrow}\}$. As before, because $A \bigcup \{\mu(\tilde{s})\} \bigcup B$ is an affinely independent set, Lemma 2 implies that $\exists \widetilde{\alpha} : \Theta \to \mathbb{R}$ such that,

$$\max_{\mu(s)\in A} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})] < \mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta})] < \min_{\mu(s)\in B} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})].$$

Define for any s,

$$l(s) := \mu(s)(\theta^*)\mathcal{G}_{s}^{\theta^*}(B)$$

Define, for any $k \in \mathbb{R}$,

$$\begin{aligned}\Delta_1(k) :=& \left((\mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(\tilde{s})) - (\max_{\mu(s)\in A} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(s)) \right), \\ \text{and } \Delta_2(k) :=& \left(\min_{\mu(s)\in B} \mathbb{E}_{\mu(s)}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(s) - (\mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta})] + kl(\tilde{s})) \right)\end{aligned}$$

Notice that $(\Delta_1(0), \Delta_2(0)) > (0, 0)$. By continuity in k, $\exists k > 0$ such that $(\Delta_1(k), \Delta_2(k)) > (0, 0)$. For any such k, choose a scalar a so that

$$\mathbb{E}_{\mu(\tilde{s})}[\widetilde{\alpha}(\boldsymbol{\theta}) + a] + kl(\tilde{s}) = 0.$$

Define

$$\alpha(\cdot) := \widetilde{\alpha}(\cdot) + a$$

and let

$$\beta(\theta) = \frac{k}{(N-1)} \mathbb{1}_{\theta=\theta^*}.$$

Now define a common-value affine coordination game Γ with $h(x) = x$, and (α, β) as above, and consider the strategy profile $\sigma = \mathbb{1}_B$. Note that we have

$$\max_{\mu(s)\in A} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + l(s)\right\} < \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + kl(\tilde{s}) = 0 < \min_{\mu(s)\in B} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + kl(s)\right\}$$

Further note that for any s,

$$\mathbb{E}_{\mu(s)}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_s^{\boldsymbol{\theta}}(B)] = kl(s).$$

Therefore the following conditions are satisfied.

$$\begin{aligned}\max_{\mu(s)\in A}\left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(s)}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_s^{\boldsymbol{\theta}}(B)]\right\} &< 0 \\ \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(\tilde{s})}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_{\tilde{s}}^{\boldsymbol{\theta}}(B)] &= 0 \\ \min_{\mu(s)\in B}\left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(s)}[\beta(\boldsymbol{\theta})(N-1)\mathcal{G}_s^{\boldsymbol{\theta}}(B)]\right\} &> 0.\end{aligned} \tag{8}$$

Equations (8) imply that the strategy profile $\sigma = \mathbb{1}_B$ constitutes an equilibrium under $\mathcal{G}$, i.e., $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{G})$. Moreover, (IC:NP) holds with equality for $\tilde{s}$, and strictly for any $s \leqslant \hat{s}$ such that $s \neq s$. Finally, notice that, since $\mathcal{F}_{\tilde{s}}^{\theta^*}(B) > \mathcal{G}_{\tilde{s}}^{\theta^*}(B)$,

$$\begin{aligned}\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \mathbb{E}_{\mu(\tilde{s})}[\beta(\boldsymbol{\theta})\mathcal{G}_{\tilde{s}}^{\boldsymbol{\theta}}(B)] &= \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + k\mu(\tilde{s})(\theta^*)\mathcal{F}_{\tilde{s}}^{\theta^*}(B) \\ &> \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + kl(\tilde{s}) = 0.\end{aligned}$$

Therefore, $\sigma \notin \overline{\mathcal{E}}(\Gamma, \mathcal{F})$, a contradiction. Thus, case 2 cannot hold either. This completes the proof of Theorem 3. □

A.4. Contour-CAD and Separable Games

In this section, we consider a special case of the setting of Section 3.4, where $\beta(\cdot)$ is independent of the state. We call such games "Separable common value affine coordination games." The net payoff of a player from taking action $a = 1$ in such a game is given by

$$d(A_{-i}, \theta) = \alpha(\theta) + \beta h(A_{-i}).$$

for some affine, increasing $h(\cdot)$ and $\beta \geqslant 0$. An analog of Theorem 3 holds. The only difference is that we do not need cCAD increase for each state, but only in expectations.

Theorem 4: *Let $\mathcal{F}$ and $\mathcal{G}$ be two distributions with identical marginal distributions.*

1. *$\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G} \implies \overline{\mathcal{E}}(\Gamma, \mathcal{F}) \supseteq \overline{\mathcal{E}}(\Gamma, \mathcal{G})$ for all separable common-value affine coordination games Γ.*
2. *Suppose $\mathcal{G}$ (and hence $\mathcal{F}$) is affinely independent. Then $\overline{\mathcal{E}}(\Gamma, \mathcal{F}) \supseteq \overline{\mathcal{E}}(\Gamma, \mathcal{G})$ for all separable common-value affine coordination games $\Gamma \implies \mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$.*

Proof of Theorem 4. The proof approach is similar. We first prove (1). Supose that $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$. Consider a separable common-value affine coordination game Γ. Let $h(x) = kx + l$ for some $k \geqslant 0$. Suppose that σ is a cutoff equilibrium under $\mathcal{G}$

with a cutoff $\tilde{s}$. Therefore,

$$\mathbb{E}\left[d(\mathbf{A}_{-\mathbf{i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{G}\right] \geqslant 0 \quad \forall s \geqslant \tilde{s}$$
$$\mathbb{E}\left[d(\mathbf{A}_{-\mathbf{i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{G}\right] \leqslant 0 \quad \forall s < \tilde{s}$$

For any s,

$$\begin{aligned}
&\mathbb{E}\left[d(\mathbf{A}_{-\mathbf{i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{F}\right] - \mathbb{E}\left[d(\mathbf{A}_{-\mathbf{i}}, \boldsymbol{\theta})|s, \sigma; \mathcal{G}\right] \\
&= \mathbb{E}\left[\alpha(\boldsymbol{\theta}) + \beta h(\mathbf{A}_{-\mathbf{i}})|s, \sigma; \mathcal{F}\right] - \left[\alpha(\boldsymbol{\theta}) + \beta h(\mathbf{A}_{-\mathbf{i}})|s, \sigma; \mathcal{G}\right] \\
&= \beta k\left(\mathbb{E}\left[\mathbf{A}_{-\mathbf{i}}|s, \sigma; \mathcal{F}\right] - \mathbb{E}\left[\mathbf{A}_{-\mathbf{i}}|s, \sigma; \mathcal{G}\right]\right) \\
&= \beta k[\mathcal{F}_s(\tilde{s}^{\uparrow}) - \mathcal{G}_s(\tilde{s}^{\uparrow})]
\end{aligned}$$

Since $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$, the above expression is non-negative if $s \geqslant \tilde{s}$ and non-positive otherwise. This implies that if σ was an equilibrium under $\mathcal{G}$, then it remains an equilibrium under $\mathcal{F}$, because (IC:P) continues to hold for all $s \geqslant \tilde{s}$, and (IC:NP) continues to hold for all $s < \tilde{s}$. Thus, $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{F})$.

Next we prove (2). Suppose that $\overline{\mathcal{E}}(\Gamma, \mathcal{F}) \supseteq \overline{\mathcal{E}}(\Gamma, \mathcal{G})$ for all separable common-value affine coordination games Γ but $\mathcal{F} \not\succcurlyeq_{cCAD} \mathcal{G}$. Then there exists $\tilde{s}, \hat{s}$ such that at least one of the following holds:

1. $\hat{s} \leqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}(\hat{s}^{\uparrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\uparrow})$, or
2. $\hat{s} \geqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}(\hat{s}^{\downarrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\downarrow})$.

Case 1: Suppose $\hat{s} \leqslant \tilde{s}$ and $\hat{s} \leqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}(\hat{s}^{\uparrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\uparrow})$

Define

$$A := \{\mu(s) : s \in \mathcal{S}, s < \hat{s}\}.$$

$$B := \{\mu(s) : s \in \mathcal{S}, s \geqslant \hat{s}, s \neq \tilde{s}\}.$$

Now $A \bigcup \{\mu(\tilde{s})\} \bigcup B$ is an affinely independent set. By Lemma 2, $\exists \lambda \in \mathbb{R}^N$ such that

$$\max_{\mu(s) \in A} \lambda' \mu(s) < \lambda' \mu(\tilde{s}) < \min_{\mu(s) \in B} \lambda' \mu(s).$$

Notice that $\lambda' \mu(\cdot) = \mathbb{E}_{\mu(\cdot)}[\lambda \boldsymbol{\theta}]$. Therefore, $\exists \lambda : \Theta \to \mathbb{R}$ such that,

$$\max_{\mu(s) \in A} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})] < \mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta})] < \min_{\mu(s) \in B} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})].$$

Define

$$\alpha(\cdot) := \lambda(\cdot) - \frac{\mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta})] + \min_{\mu(s) \in B} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})]}{2}.$$

Notice that $\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] = \dfrac{\mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta})] - \min_{\mu(s) \in B} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})]}{2} < 0$. Therefore, we have

$$\max_{\mu(s) \in A} \mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] < \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] < 0 < \min_{\mu(s) \in B} \mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})]. \tag{9}$$

Define

$$\beta := -\frac{\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})]}{\mathcal{G}_{\tilde{s}}(\hat{s}^{\uparrow})}.$$

Notice that $\beta > 0$. Also, (9) and the monotonicity of $\mathcal{G}_s(\hat{s}^{\uparrow})$ (in s) imply,

$$\begin{aligned}
\max_{\mu(s)\in A} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{G}_s(\hat{s}^{\uparrow})\right\} &< 0 \\
\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{G}_{\tilde{s}}(\hat{s}^{\uparrow}) &= 0 \\
\min_{\mu(s)\in B} \left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{G}_s(\hat{s}^{\uparrow})\right\} &> 0.
\end{aligned} \tag{10}$$

Notice that it is possible that $\mathcal{G}_s(\hat{s}^{\uparrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\uparrow})$. However, the last inequality follows from $\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] < 0 < \min_{\mu(s)} \mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})]$ and $\mathcal{G}_s(\hat{s}^{\uparrow}) \geqslant 0$.

Consider a separable common-value affine coordination game Γ with $h(x) = x$, and (α, β) defined above. Consider a strategy profile σ such that $P(\sigma) = \hat{s}^{\uparrow}$ and $NP(\sigma) = \mathcal{S}\backslash\hat{s}^{\uparrow}$. Then, Equations (10) simply say mean that this strategy constitues an equilibrium under $\mathcal{G}$, i.e., $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{G})$. Moreover, (IC:P) holds with an equality for $\tilde{s}$, and holds strictly for any $s \geqslant \hat{s}$ such that $s \neq \tilde{s}$. Finally, notice that

$$\mathcal{F}_{\tilde{s}}(\hat{s}^{\uparrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\uparrow}) \implies \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{F}_{\tilde{s}}(\hat{s}^{\uparrow}) < 0.$$

Therefore, $\sigma \notin \overline{\mathcal{E}}(\Gamma, \mathcal{F})$, a contradiction. Thus, case 1 cannot hold.

Case 2: $\hat{s} \geqslant \tilde{s}$ and $\mathcal{F}_{\tilde{s}}(\hat{s}^{\downarrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\downarrow})$:

Define $A := \{\mu(s) : s \in \mathcal{S}, s \leqslant \hat{s}, , s \neq \tilde{s}\}$ and $B := \{\mu(s) : s \in \mathcal{S}, s > \hat{s}\}$. Notice that $\mathcal{F}_{\tilde{s}}(\hat{s}^{\downarrow}) < \mathcal{G}_{\tilde{s}}(\hat{s}^{\downarrow}) \iff \mathcal{F}_{\tilde{s}}(B) > \mathcal{G}_{\tilde{s}}(B)$ since $B = \mathcal{S}\backslash\{\hat{s}^{\downarrow}\}$.

Proceeding as before, we use the fact that $A\bigcup\{\mu(\tilde{s})\}\bigcup B$ is an affinely independent set. Therefore, by Lemma 2, $\exists\lambda : \Theta \to \mathbb{R}$ such that,

$$\max_{\mu(s)\in A} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})] < \mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta})] < 0 < \min_{\mu(s)\in B} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})].$$

Let $a > 0$ be such that, for all $s \leqslant \hat{s}, s \neq \tilde{s}$,

$$a\left[\mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta}) - \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})]\right] > \max_{s\leqslant\hat{s}}(\mathcal{G}_s(B) - \mathcal{G}_{\tilde{s}}(B)).$$

Such an a exists since $\mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta})] > \max_{\mu(s)\in A} \mathbb{E}_{\mu(s)}[\lambda(\boldsymbol{\theta})]$. Using this a, define $\alpha(\cdot) = a\lambda(\cdot)+b$ where $b := -(a\mathbb{E}_{\mu(\tilde{s})}[\lambda(\boldsymbol{\theta})]+\mathcal{G}_{\tilde{s}}(B))$. By definition, $\mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})]+\mathcal{G}_{\tilde{s}}(B) = 0$. By the choice of a, for all $s \in A$,

$$0 = \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \mathcal{G}_{\tilde{s}}(B) > \mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \mathcal{G}_s(B).$$

Finally, by the monotonicity of $\mathcal{G}_s(B)$ (in s), for all $s \in B$,

$$0 = \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \mathcal{G}_{\tilde{s}}(B) < \mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \mathcal{G}_s(B)$$

Therefore, by letting $\beta := 1$, we have,

$$\begin{aligned}\max_{\mu(s)\in A}\left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{G}_s(B)\right\} <&0\\ \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{G}_{\tilde{s}}(B) =&0\\ \min_{\mu(s)\in B}\left\{\mathbb{E}_{\mu(s)}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{G}_s(B)\right\} >&0.\end{aligned} \tag{11}$$

Similar to before, consider a game Γ with $h(x) = x$, and (α, β) as obtained above. Notice that the equations (11) are simply (IC:P) and (IC:NP) for the strategy profile $\sigma = \mathbb{1}_B$. Thus, $\sigma \in \overline{\mathcal{E}}(\Gamma, \mathcal{G})$. Moreover, (IC:NP) holds with an equality for $\tilde{s}$, and holds strictly for any $s \leqslant \hat{s}$ such that $s \neq s$. Finally, notice that

$$\mathcal{F}_{\tilde{s}}(B) > \mathcal{G}_{\tilde{s}}(B) \implies \mathbb{E}_{\mu(\tilde{s})}[\alpha(\boldsymbol{\theta})] + \beta\mathcal{F}_{\tilde{s}}(B) > 0.$$

Therefore, $\sigma \notin \overline{\mathcal{E}}(\Gamma, \mathcal{F})$, a contradiction. Thus, case 2 cannot hold either. This completes the proof of Theorem 4. □

A.5. Proof of Proposition 2

Proof. Using a similar reasoning as in Theorem 1, we can see that if $\beta(s) \leqslant 0$ for all $s \in \mathcal{S}$, then for strategy profile σ, the inequalities in Equation (2) would be reversed, i.e.,
For $\beta(s) \leqslant 0$,

$$\begin{aligned}\mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{F}] - \mathbb{E}[d(\mathbf{A_{-i}}, s)|s, \sigma; \mathcal{G}] &\leqslant 0 \quad \text{if } s \in P(\sigma)\\ &\geqslant 0 \quad \text{if } s \in NP(\sigma)\end{aligned}$$

Therefore, if $\sigma \in \mathcal{E}(\mathcal{F}, \Gamma)$, then (IC:P) continues to hold for all $s \in P(\sigma)$ under $\mathcal{G}$ and (IC:NP) holds for all $s \in NP(\sigma)$ under $\mathcal{G}$. Therefore, $\sigma \in \mathcal{E}(\Gamma, \mathcal{G})$.

Suppose that $\mathcal{E}(\Gamma, \mathcal{G}) \supseteq \mathcal{E}(\Gamma, \mathcal{F})$ for all games Γ with $\beta(s) \leqslant 0$ for all $s \in \mathcal{S}$, but $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$. Using Lemma 1 again, we know that $\exists s^*$ and $K \ni s^*$ such that,

$$\mathcal{F}(\mathbf{X}_j \in K|\mathbf{X}_i = s^*) < \mathcal{G}(\mathbf{X}_j \in K|\mathbf{X}_i = s^*).$$

Now, we define a σ with $P(\sigma) = K$, and a game Γ as follows:

$$\begin{aligned}
\alpha(s) &= 1 \\
h(A_{-i}) &= A_{-i} \\
\beta(s) & \begin{cases} = 0 & \text{if } s \notin K \\ = \dfrac{-1}{\mathbb{E}\left[\mathbf{A_{-i}} \middle| s^*; \mathcal{G}\right]} & \text{if } s \in K \text{ and } s = s^* \\ \geqslant \dfrac{-1}{\mathbb{E}\left[\mathbf{A_{-i}} \middle| s; \mathcal{G}\right]} & \text{if } s \in K \text{ and } s \neq s^*. \end{cases}
\end{aligned}$$

It is straightforward to verify that $\sigma \in \mathcal{E}(\Gamma, \mathcal{F})$, but $\sigma \notin \mathcal{E}(\Gamma, \mathcal{G})$, a contradiction.

This proves the first part of the proposition (wCAD equivalence). The second part (sCAD equivalence) makes a the same modification to the reasoning in Theorem 2, and hence omitted. □

A.6. Proof of Proposition 3

Proof. Following Meyer (1990), we define an elementary transformation on identical intervals (ETI) as follows: Given any $s < s'$, we increase the probabilities of (s, s) and (s', s') by some $\alpha > 0$, while the probabilities of (s', s) and (s, s') are decreased by α each. Proposition 1 of Meyer (1990) says that $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$ if and only if $\mathcal{F}$ can be obtained by a finite sequence of ETIs starting from $\mathcal{G}$. We now establish that any ETI increases the expected revenue. To this end, fix $s < s'$ and consider an ETI involving s and s' to construct $\widehat{\mathcal{G}}$ from $\mathcal{G}$.

Notice that the price of the object is s whenever the valuations are $(s, s), (s, s')$ or (s', s). Therefore,

$$R(\widehat{\mathcal{G}}) - R(\mathcal{G}) = \alpha(s' + s) - 2\alpha(s) = \alpha(s' - s) > 0.$$

Hence, $R(\mathcal{F}) \geqslant R(\mathcal{G})$ with the inequality being strict whenever $\mathcal{F} \neq \mathcal{G}$. □

A.7. Proof of Proposition 4

Proof. The conditional belief about the other player's type

$$f(\mathbf{X}_j = s_j | \mathbf{X}_i = s_i) = \frac{\int \pi(\theta) f(\varepsilon_j = s_j - \theta, \varepsilon_i = s_i - \theta) d\theta}{\int\int \pi(\theta) f(\varepsilon_j = s_j - \theta, \varepsilon_i = s_i - \theta) ds_j d\theta}$$

where $\pi(\theta)$ is the prior and f is the joint density of noise. Since the prior is uninformative $\pi(\theta)$ is a constant for any θ and hence factored out, which makes the denominator equal to 1. Therefore,

$$f(\mathbf{X}_j = s_j | \mathbf{X}_i = s_i) = \int f(\varepsilon_j = s_j - \theta, \varepsilon_i = s_i - \theta) d\theta = \psi(\varepsilon_j - \varepsilon_i = s_j - s_i).$$

Since f is bi-variate normal $(\varepsilon_j - \varepsilon_i) \sim N(0, 2\sigma^2(1-\rho))$. This means

$$f(\mathbf{X}_j \leqslant s_j | \mathbf{X}_i = s_i) = \Phi\left(\frac{s_j - s_i}{\sigma\sqrt{2(1-\rho)}}\right)$$

where $\Phi(\cdot)$ is the standard normal CDF. Note that if $s_j > s_i$, then a higher ρ increases this conditional belief and vice versa, which means a contour CAD increase. Note that the payoff specification in (5) with β being independent of θ is a special case of our payoff specification in Section 3.4. The equilibrium set expansion then follows from Theorem 4. □

It is worth highlighting that Proposition 4 is not true if the prior is informative (not improper). In this case, a higher ρ does not necessarily increase the similarity of information in the cCAD sense conditional on the state θ. To see this, note that $f(\mathbf{X}_j \leqslant s_j | \mathbf{X}_i = s_i, \boldsymbol{\theta} = \theta) = Prob(\varepsilon_j \leqslant s_j - \theta | \varepsilon_i = s_i - \theta, \theta)$. Since $(\varepsilon_i, \varepsilon_j)$ is bi-variate normal, the conditional distribution $\varepsilon_j | \varepsilon_i \sim N(\rho\varepsilon_i, \sigma\sqrt{1-\rho^2})$. Therefore,

$$f(\mathbf{X}_j \leqslant s_j | \mathbf{X}_i = s_i, \boldsymbol{\theta} = \theta) = \Phi\left(\frac{s_j - \theta - \rho(s_i - \theta)}{\sigma\sqrt{1-\rho}}\right) = \Phi\left(\frac{s_j - (\rho s_i + (1-\rho)\theta)}{\sigma\sqrt{1-\rho}}\right).$$

It is straightforward to verify that for $s_i = s > \theta$ and $s_j = s + z$ for some small but positive z, the above conditional belief is decreasing in ρ. Therefore, conditional on θ, increasing ρ does not increase similarly of information is cCAD sense for all θ. Accordingly, the set of equilibria may not expand for games in which the payoff difference can be represented as $d(\theta, a_j) = \alpha(\theta) + \beta(\theta) \cdot h(a_j)$, where $\beta(\theta)$ is not a constant; that is, Theorem 3 does not hold.

A.8. Non-exchangeable signal distributions

The Game Γ and Information Structure $\mathcal{F}$

There are N players, indexed by i or $j \neq i$. They simultaneously and independently choose whether to act ($a_i = 1$) or not ($a_i = 0$). Each player i's payoff depends on the *weighted* aggregate action by others. That is, $\exists \vec{\lambda} \in \mathbb{R}_+^N$ such that

$$\mathbf{A}_{-\mathbf{i}} = \sum_{j \neq i} \lambda_j a_j.$$

Rest of the setup remains the same as in the main model. Let $\mathcal{F}_s^i(\cdot) \in \Delta(\mathcal{S}^{N-1})$ denote the conditional distribution of player i over other players' types, given that a player i has type s. Unlike the main text, $\mathcal{F}$ need not be exchangeable, this conditional distribution is indexed with the player identity. For any pair of agents i and j, and any $T \subset \mathcal{S}$, we define

$$\mathcal{F}_s^i(\mathbf{X}_j \in T) := Prob(\mathbf{X}_j \in T | \mathbf{X}_i = s).$$

Suppose that $\vec{s} = (s_1, s_2, \ldots s_N) \in \mathcal{S}^N$ is the realized type profile. A player i with type $\mathbf{X}_i = s$ get a payoff of $u(a_i = 1, \mathbf{A_{-i}}, s)$ if she acts and $u(a_i = 0, \mathbf{A_{-i}}, s)$ if she does not act. The net payoff from taking action

$$d(\mathbf{A_{-i}}, s) := u(a_i = 1, \mathbf{A_{-i}}, s) - u(a_i = 0, \mathbf{A_{-i}}, s) = \alpha(s) + \beta(s) h(\mathbf{A_{-i}}),$$

where $h(\cdot)$ is increasing.

Definition 10: *The game Γ exhibits **strategic complementarity** if $\beta(\cdot) \geqslant 0$ and **strategic substitutability** if $\beta(\cdot) \leqslant 0$ for all $s \in \mathcal{S}$. The game is said to be **affine** if $h(\cdot)$ is affine.*

Definition 11 (Weak Concentration along a Diagonal): *We say that $\mathcal{F}$ has a "higher concentration along a diagonal" than $\mathcal{G}$, or $\mathcal{F}$ **is wCAD higher than** $\mathcal{G}$, denoted by $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$, if,*

1. *$\mathcal{F}$ and $\mathcal{G}$ have the same marginal distributions. And,*
2. *For all $i, j \in \mathbf{N}, s \in \mathcal{S}$, and any pair of agents i, j*
   (a) *$\mathcal{F}_s^i(\mathbf{X}_j = s) \geqslant \mathcal{G}_s(\mathbf{X}_j = s)$, and*
   (b) *$\mathcal{F}_s(\mathbf{X}_j = s') \leqslant \mathcal{G}_s(\mathbf{X}_j = s')$ whenever $s' \neq s$.*

If $\mathbf{X} \sim \mathcal{F}$ and $\mathbf{Y} \sim \mathcal{G}$ with $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$, we say $\mathbf{X} \succcurlyeq_{wCAD} \mathbf{Y}$, i.e., we use $\mathbf{X} \succcurlyeq_{wCAD} \mathbf{Y}$ and $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$ interchangeably. A similar result as Lemma **??** holds when we relax exchangeability.

Lemma 3: *Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{S}^N$-valued random variables with distributions $\mathcal{F}$ and $\mathcal{G}$ respectively. Moreover, $\mathcal{F}$ and $\mathcal{G}$ have identical marginals. Then, the following are equivalent.*

1. *$\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$.*
2. *For all $s \in \mathcal{S}$ and $K \subseteq \mathcal{S}$ such that $s \in K$, and $\vec{\lambda} \in \mathbb{R}_+^N$,*

$$\mathbb{E}\left[\sum_{j \neq i} \lambda_j \mathbb{1}_{\mathbf{X}_j \in K} \middle| \mathbf{X}_i = s\right] \geqslant \mathbb{E}\left[\sum_{j \neq i} \lambda_j \mathbb{1}_{\mathbf{Y}_j \in K} \middle| \mathbf{Y}_i = s\right].$$

Proof of Lemma 3. We establish this for some i such that $\mathbf{X}_i = s$. Then,

$$\begin{aligned}
\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G} &\iff \mathcal{F}_s^i(\mathbf{X}_j \in K \middle| \mathbf{X}_i = s) \geqslant \mathcal{F}_s^i(\mathbf{Y}_j \in K \middle| \mathbf{Y}_i = s) \quad \forall i, j, s, K \ni s, \\
&\iff \mathbb{E}\left[\lambda_j \mathbb{1}_{\mathbf{X}_j \in K} \middle| \mathbf{X}_i = s\right] \geqslant \mathbb{E}\left[\lambda_j \mathbb{1}_{\mathbf{Y}_j \in K} \middle| \mathbf{Y}_i = s\right] \quad \forall s, K \ni s. \\
&\implies \mathbb{E}\left[\sum_{j \neq i} \lambda_j \mathbb{1}_{\mathbf{X}_j \in K} \middle| \mathbf{X}_i = s\right] \geqslant \mathbb{E}\left[\sum_{j \neq i} \lambda_j \mathbb{1}_{\mathbf{Y}_j \in K} \middle| \mathbf{Y}_i = s\right] \quad \forall s, K \ni s.
\end{aligned}$$

For the reverse, suppose that $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$. Therefore, $\exists i, j, s, K \ni s$, such that

$$\mathcal{F}_s^i(\mathbf{X}_j \in K) < \mathcal{G}_s^i(\mathbf{X}_j \in K).$$

Then, let $\vec{\lambda}$ be defined by $\lambda_k = \mathbb{1}_{k \in \{i,j\}}$. Then,

$$\mathbb{E}\left[\sum_{j \neq i} \lambda_j \mathbb{1}_{\mathbf{X}_j \in K} \middle| \mathbf{X}_i = s\right] < \mathbb{E}\left[\sum_{j \neq i} \lambda_j \mathbb{1}_{\mathbf{Y}_j \in K} \middle| \mathbf{Y}_i = s\right]$$

contradicting the hypothesis. □

Theorem 5: *Let $\mathcal{F}$ and $\mathcal{G}$ be two joint distributions over $\mathcal{S}^N$ and $h(\cdot)$ is affine and increasing. The following are equivalent.*

1. $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$.
2. $\mathcal{E}(\Gamma, \mathcal{F}) \supseteq \mathcal{E}(\Gamma, \mathcal{G})$ *for all* Γ *that exhibits strategic complementarity.*
3. $\mathcal{E}(\Gamma, \mathcal{F}) \subseteq \mathcal{E}(\Gamma, \mathcal{G})$ *for all* Γ *that exhibits strategic substitutability.*

Proof of Theorem 5. The proof of (1) $\implies$ (2), (3) is identical to the proof of Theorem 1 with the only difference being that the conditional distributions now have to be indexed by the player identity i. Therefore, we omit it.

Now, we show (2) $\implies$ (1) and (3) $\implies$ (1). Suppose that $\mathcal{E}(\Gamma, \mathcal{F}) \supseteq \mathcal{E}(\Gamma, \mathcal{G})$ for all Γ with $\beta(s) \geqslant 0$ for all $s \in \mathcal{S}$, but $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$. Therefore, $\exists i, j$ a $s \in \mathcal{S}$, and $K \subseteq \mathcal{S}$ with $s \in K$, such that, $\mathcal{F}_s^i(\mathbf{X}_j \in K) < \mathcal{G}_s^i(\mathbf{X}_j \in K)$. Let $\vec{\lambda}$ be defined by $\lambda_k = \mathbb{1}_{k \in \{i,j\}}$. Then, this game is essentially a 2−player symmetric game. Therefore, replicating the construction from the proof of Theorem 1 with $N = 2$ and using Lemma 3, we obtain the desired equivalence. □

A.9. Relationship of CAD with other orders

Below, we discuss how our proposed orders—strong CAD, weak CAD, and contour CAD—relate to well-known orders like the supermodular order or the concordance order. In two dimensions, the supermodular and the concordance orders are equivalent (see Meyer and Strulovici, 2012). For more than two dimensions, the supermodular order is strictly stronger than the concorance order, i.e., $\mathcal{F} \succcurlyeq_{sm} \mathcal{G} \implies \mathcal{F} \succcurlyeq_{conc} \mathcal{G}$, where $\succcurlyeq_{sm}$ and $\succcurlyeq_{conc}$ denote the supermodular and the concordance orders respectively.

Proposition 5: *Suppose $\mathcal{F}$ and $\mathcal{G}$ are joint distributions over $\mathcal{S}^N$-valued, exchangeable random variables.*

1. *For $N = 2$, $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G} \implies \mathcal{F} \succcurlyeq_{sm} \mathcal{G}$. But $\exists$ $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F} \succcurlyeq_{sm} \mathcal{G}$ but $\mathcal{F} \not\succcurlyeq_{cCAD} \mathcal{G}$.*
2. *For $N > 2$, $\succcurlyeq_{wCAD}$ and $\succcurlyeq_{sm}$ are not nested. Also, $\succcurlyeq_{cCAD}$ and $\succcurlyeq_{sm}$ are not nested.*

Proof of Proposition 5. Proposition 1 established that $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G} \implies \mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$. We first prove part (1). Suppose that $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$. Consider any $s, s' \in \mathcal{S}$. Let $s' \geqslant s$ wlog. Then,

$$\begin{aligned}\mathcal{F}(\mathbf{X}_1 \leqslant s, \mathbf{X}_2 \leqslant s') &= \sum_{\hat{s} \leqslant s} \mathcal{F}_{\hat{s}}(\mathbf{X}_2 \leqslant s')\mathcal{F}(\mathbf{X}_1 = \hat{s}) \\ &\geqslant \sum_{\hat{s} \leqslant s} \mathcal{G}_{\hat{s}}(\mathbf{X}_2 \leqslant s')\mathcal{G}(\mathbf{X}_1 = \hat{s}) \\ &= \mathcal{G}(\mathbf{X}_1 \leqslant s, \mathbf{X}_2 \leqslant s')\end{aligned}$$

where the inequality is due to the fact that $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$ and the marginal distributions of $\mathcal{F}$ and $\mathcal{G}$ coincide. Similarly,

$$\begin{aligned}\mathcal{F}(\mathbf{X}_1 \geqslant s, \mathbf{X}_2 \geqslant s') &= \sum_{\hat{s} \geqslant s'} \mathcal{F}_{\hat{s}}(\mathbf{X}_1 \geqslant s)\mathcal{F}(\mathbf{X}_2 = \hat{s}) \\ &\geqslant \sum_{\hat{s} \geqslant s'} \mathcal{G}_{\hat{s}}(\mathbf{X}_1 \geqslant s)\mathcal{G}(\mathbf{X}_2 = \hat{s}) \\ &= \mathcal{G}(\mathbf{X}_1 \leqslant s, \mathbf{X}_2 \leqslant s')\end{aligned}$$

where, again, the inequality is due to the fact that $\mathcal{F} \succcurlyeq_{cCAD} \mathcal{G}$ and the marginal distributions of $\mathcal{F}$ and $\mathcal{G}$ coincide. Therefore, $\mathcal{F} \succcurlyeq_{sm} \mathcal{G}$, and hence, $\mathcal{F} \succcurlyeq_{conc} \mathcal{G}$.

For more than two dimensions, consider two $\{0,1\}^3$-valued distributions, $\mathcal{F}$ and $\mathcal{G}$, in Table 1.[20]

s_1	s_2	s_3	$\mathcal{F}$	$\mathcal{G}$
0	0	0	$\frac{1}{3}$	0
0	0	1	0	$\frac{1}{4}$
0	1	0	0	$\frac{1}{4}$
1	0	0	0	$\frac{1}{4}$
0	1	1	$\frac{1}{6}$	0
1	0	1	$\frac{1}{6}$	0
1	1	0	$\frac{1}{6}$	0
1	1	1	$\frac{1}{6}$	$\frac{1}{4}$

Table 1: $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$ but $\mathcal{F}$ and $\mathcal{G}$ not $\succcurlyeq_{sm}$-ranked

Let us first show that $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$ but $\mathcal{F} \not\succcurlyeq_{sm} \mathcal{G}$. Notice that, for any i, j such that $i \neq j$ and $a \in \{0,1\}$, $\mathcal{F}(\mathbf{X}_i = a|\mathbf{X}_j = a) = \frac{1}{3} > \frac{1}{4} = \mathcal{G}(\mathbf{X}_i = a|\mathbf{X}_j = a)$. Therefore, $\mathcal{F} \succcurlyeq_{wCAD} \mathcal{G}$. Consider the supermodular function $f(x) = \mathbb{1}_{x=(1,1,1)}$. Then, $\mathbb{E}_{\mathcal{F}}[f(\mathbf{X})] < \mathbb{E}_{\mathcal{G}}[f(\mathbf{X})]$. Also, consider $g(x) = \mathbb{1}_{x=(0,0,0)}$. Then, $\mathbb{E}_{\mathcal{F}}[g(\mathbf{X})] > \mathbb{E}_{\mathcal{G}}[g(\mathbf{X})]$. Therefore, $\mathcal{F} \not\succcurlyeq_{sm} \mathcal{G}$.

[20] This example is due to Margaret Meyer and Bruno Strulovici, and can be found here.

To show that the orders are not nested, however, we also need to show that there are exchangeable distributions $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F} \succcurlyeq_{sm} \mathcal{G}$ but $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$ for more than 2 dimensions. The example below, in the spirit of the one in Figure 1, establishes this. Let $\mathbf{X}$ be a $\{1,2,3\}^3-$valued random variable. Suppose that $\mathcal{G}$ is the joint distribution of $\mathbf{X}$ with $\mathbf{X}_i$'s being independent and uniformly distributed. Now, consider the following operations for some small $\alpha > 0$ (so that probabilities stay non-negative):

- If the realization is $(2,2,2)$, reduce the mass by 6α.
- If the realization is $(2,1,2)$ or any of its permutations, then increase the mass by 2α.
- If the realization is $(2,2,3)$ or any of its permutations, then increase the mass by 2α.
- If the realization is $(1,2,3)$ or any of its permutation, then reduce the mass by α.

It is easy to check that $\mathcal{F}(\mathbf{X}_2 = 2|\mathbf{X}_1 = 2) < \mathcal{G}(\mathbf{X}_2 = 2|\mathbf{X}_1 = 2)$. Therefore, $\mathcal{F} \not\succcurlyeq_{wCAD} \mathcal{G}$. In fact, $\mathcal{F} \not\succcurlyeq_{cCAD} \mathcal{G}$ too since

$$\mathcal{F}(\mathbf{X}_2 \in \{1,2\}|\mathbf{X}_1 = 2) < \mathcal{G}(\mathbf{X}_2 \in \{1,2\}|\mathbf{X}_1 = 2).$$

Finally, to check that $\mathcal{F} \succcurlyeq_{sm} \mathcal{G}$ necessitates proving that $\mathcal{F}$ can be obtained from $\mathcal{G}$ through "elementary transformations" as defined in Meyer and Strulovici (2015). This is indeed the case, however, we omit the details here.

□

B. Online Appendix

B.1. Common learning and similarity of information

Awaya and Krishna (2025) show how more correlation in players' signals impedes common learning thereby hindering coordination. We briefly discuss their leading example here to illustrate the key contrast.

There is a binary state of the world $\theta \in \{G, B\}$. There are two players, and two periods. In each period, each agent receives some $\{0, 1\}$-valued signal about the state. Signal realization of 1 is conclusive of state G while 0 can arise in both states. Signals are independent across periods but can be correlated across players within a given period. At the end of period 2, they decide simultaneously and privately whether to invest at a cost c or not. When $\theta = B$, investment yields no returns, while in state G, an investment yields a return of 1 if the other player invests. Thus, both players would like to coordinate to invest in state G and not invest in state B.

They point out that when the signals are conditionally independent across players, there are cost parameters and signal structures such that investing upon receiving at least one 1 in two periods is an equilibrium. Subsequently, for some such cost and signal structures, they do the analogous exercise as ours: they keep the marginal distribution unchanged and make the signals more similar (in the wCAD or PQD order) within a period. Yet, the unique equilibrium is that no player invests regardless of the number of 1s they receive!

The source of this finding is that increasing similarity *within a period* does not imply that the vector of signals across the two periods becomes more similar. In particular, it is possible that when a player receives exactly one 1 and one 0, he assigns a *higher* probability to the other agent receiving both zeroes with signals being more similar within a period. That is, we can view the two players' signals across two periods as a $\{(0,0), (0,1), (1,0), (1,1)\}^2$-valued random variable. Then, $Prob(\mathbf{X}_2 \in \{(0,1), (1,0), (1,1)\} | \mathbf{X}_1 = (0,1))$ may decrease with signals being more similar within a period. Here, $\mathbf{X}_i$ is player i's signal in two periods.

B.2. Information similarity and rationalizability

In this section, we explore the effect of increases in similarity in the CAD orders on the set of rationalizable actions (rather than the set of equilibria). Consider the example discussed in the introduction. We show below that any action that is rationalizable for a type continues to be rationalizable if information becomes more similar in the sense of CAD.

Let $\mathcal{S}$ be the set of types and $\mathcal{G}$ be the joint distribution of types. A mapping $x : \mathcal{S} \to \mathbb{R}$ describes the payoff parameter of interest to a player. Player i knows her own payoff parameter $x(s_i)$ but may not know the other player's payoff parameter $x(s_j)$.

$$\mathcal{G}_s(T) := Prob(\mathbf{X}_j \in T | \mathbf{X}_i = s)$$

describes player i of type s's belief about the other player's type. Each player decide whether or not to invest, and payoffs are as follows.

	Invest	Don't
Invest	$x(s_1), x(s_2)$	$x(s_1) - 1, 0$
Don't	$0, x(s_2) - 1$	$0, 0$

Below we show that if an action is rationalizable, then it continues to be so when the types become more similar in the sense of increasing wCAD (or sCAD).

Proposition 6: *If $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$, then any action a_i that is rationalizable for a type s_i of player i under $\mathcal{G}$ remains rationalizable under $\mathcal{F}$.*

Proof. Consider $\phi_i : \mathcal{S} \to \mathbb{R}$. Given the information structure $\mathcal{G}$, an event $E = E_1 \times E_2$ for $E_i \subseteq \mathcal{S}$ is said to be $(\phi_1, \phi_2)-$ believed if each player i of type s_i believes that the probability of the event E is at least $\phi_i(s_i)$. Formally, define

$$B_i^{\phi_i}(E|\mathcal{G}) := \{(s_1, s_2) | s_i \in E_i, \mathcal{G}_{s_i}(E_j) \geqslant \phi_i(s_i)\}.$$

Then, the set of states where the event E is $(\phi_1, \phi_2)-$believed is

$$B_*^{\phi_1,\phi_2}(E|\mathcal{G}) = B_1^{\phi_1}(E|\mathcal{G}) \bigcap B_2^{\phi_2}(E|\mathcal{G})$$

There is a common $(\phi_1, \phi_2)-$ belief of the event E if it is $(\phi_1, \phi_2)-$ believed that it is $(\phi_1, \phi_2)-$ believed, and so on. The set of states where the event E is common $(\phi_1, \phi_2)-$ believed is

$$C^{\phi_1,\phi_2}(E|\mathcal{G}) := \bigcap_{n \geqslant 1} [B_*^{\phi_1,\phi_2}]^n (E|\mathcal{G})$$

It follows from Proposition 1 in Morris et al. (2016) that under the information structure $\mathcal{G}$, for any type s_i of player i, the action invest is rationalizable if $s_i \in C_i^{1-x,1-x}(T|\mathcal{G})$ and the action not invest is rationalizable if $s_i \in C_i^{x,x}(T|\mathcal{G})$. The argument is as follows.

Let $R_i^1(\mathcal{G})$ be the set of types such that investment is level 1 rationalizable for player i. If $x(s_i) < 0$, player i gets a strictly higher payoff from not investing regardless of what the opponent does. Note that $\mathcal{G}_{s_i}(\mathcal{S}) \geqslant 1 - x(s_i)$ iff $x(s_i) \geqslant 0$ (since player i assigns probability 1 to event $\mathcal{S}$). Therefore,

$$R_i^1(\mathcal{G}) = B_i^{1-x}(\mathcal{S}|\mathcal{G}).$$

The action invest is rational for both players when $\mathcal{S}^2$ is $(1-x, 1-x)-$ believed. We write this set of types as $B_*^{1-x,1-x}(\mathcal{S}^2|G) = B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_1 \times B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_2$.

Let $R_i^2(\mathcal{G})$ be the set of types for which investment is level 2 rationalizable. This is the set of types $s_i \in B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_i$ such that $\mathcal{G}_{s_i}(B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_j) \geqslant 1-x(s_i)$. Since player j will not invest if $s_j \notin B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_j$, the payoff from invest is at most $x(s_i) - (1 - \mathcal{G}_{s_i}(B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_j))$. Therefore, if

$$\mathcal{G}_{s_i}(B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})_j) < 1 - x(s_i),$$

player i of type s_i strictly prefers not invest over invest. Therefore,

$$R_i^2(\mathcal{G}) = B_i^{1-x}(B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})).$$

Iterating this argument, we get

$$R_i^{k+1}(\mathcal{G}) = B_i^{1-x}([B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})).$$

The action invest is rationalizable for both players if it is $k-$th level rationalizable for all k. In other words, exactly when $\mathcal{S}^2$ is common $(1-x, 1-x)-$ believed:

$$R^\infty(\mathcal{G}) = C^{1-x,1-x}(\mathcal{S}^2|G).$$

By a symmetric argument, action not invest is rationalizable exactly when $\mathcal{S}^2$ is common $(x, x)-$ believed.

Note that for any information structure $\mathcal{G}$, $[B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})_1 = [B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})_2$ for any $k = 1, 2, \ldots \infty$. By definition, $F \geqslant_{wCAD} G$ implies for any $s_i \in [B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})_i$,

$$\mathcal{F}_{s_i}([B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})_j) \geqslant \mathcal{G}_{s_i}([B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})_j)$$

Therefore,

$$\begin{gathered}
B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G}) = B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{F}) \\
B_i^{1-x}(B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{G})) \subseteq B_i^{1-x}(B_*^{1-x,1-x}(\mathcal{S}^2|\mathcal{F})) \\
\ldots \\
B_i^{1-x}([B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G})) \subseteq B_i^{1-x}([B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{F})) \\
\implies \bigcap_{k \geqslant 1} [B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{G}) \subseteq \bigcap_{k \geqslant 1} [B_*^{1-x,1-x}]^k(\mathcal{S}^2|\mathcal{F}) \\
\text{or, } C^{1-x,1-x}(\mathcal{S}^2|\mathcal{G}) \subseteq C^{1-x,1-x}(\mathcal{S}^2|\mathcal{F}) \\
\text{or, } R^\infty(\mathcal{G}) \subseteq R^\infty(\mathcal{F}).
\end{gathered}$$

This means if the action invest is rationalizable for a player i of type s_i under information structure $\mathcal{G}$, then it remains rationalizable under $\mathcal{F} \geqslant_{wCAD} \mathcal{G}$. Analogous arguments can be made for the action "not invest" as well. □